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\# The flux freezing condition is a direct consequence of the induction equation in an ideal Magnetohydrodynamic setup.

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B})$$

Assuming that time taken for redistribution of $\vec{u}$, $\vec{B}$ field is much faster than the time taken to move the magnets, we claim that the system goes through several equilibrium states quasistatically.

Thus, $\frac{\partial \vec{B}}{\partial t} = 0 = \vec{\nabla} \times (\vec{u} \times \vec{B})$.

Which is trivially solved by $\vec{u} \times \vec{B} = 0$.

Additionally in the rest frame of the fluid parcel, $\vec{J} = \sigma \vec{E}$.

For a perfect conductor, $\vec{E}$ must be zero in the rest frame of the fluid parcel.

This implies that $\vec{u}$ must not be perpendicular to $\vec{B}$, as that would induce an electric field leading to infinite currents.

$$\therefore \quad \vec{u} \parallel \vec{B}$$

$u_{\parallel} \rightarrow$ velocity parallel to $\vec{B}$

$u_{\perp} \rightarrow$ velocity perpendicular to $\vec{B}$

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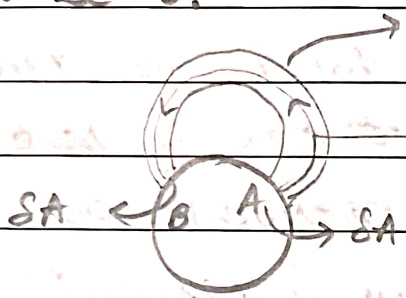
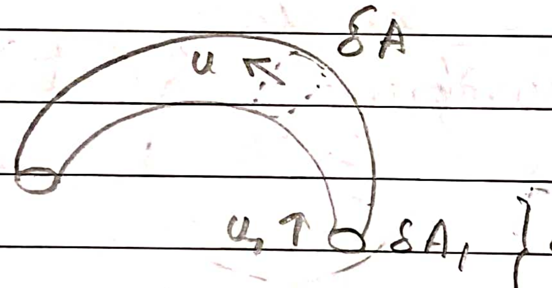
Claim: The velocity field along each field line originating and terminating on a magnet must be 0.



Since $u_{\perp} = 0$, we can model this surface like a pipe.



We write the continuity eqn. for an incompressible fluid →

$$u_{\parallel} \delta A = u_1 \delta A_1 = \text{rate of change of flux tube volume.}$$

But from boundary condition of fluid on the magnet's surface, $u_1 = 0$.

$\therefore$ VOLUME OF EVERY FLUX TUBE IS CONSTANT.

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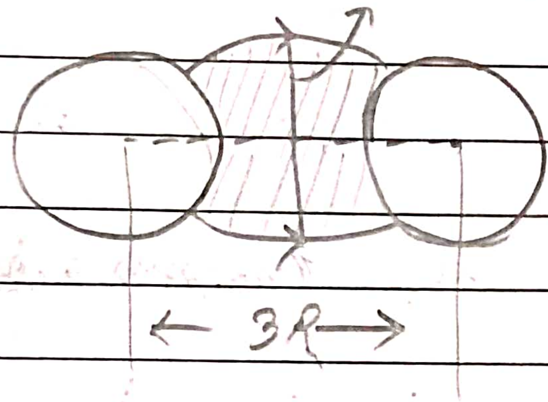
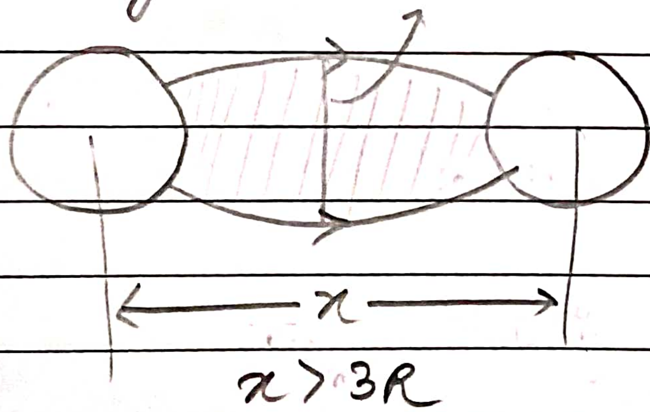
Let us consider a flux tube, which is axisymmetric starting and termination from one magnet to other.

cross section = A_0 (say)



On stretching →

cross section = A



$x > 3R$

To conserve the volume enclosed, the cross section of the tubes decrease. But by definition they have the same flux through them as before, →

SO THE MAGNETIC FIELD STRENGTH INCREASES.

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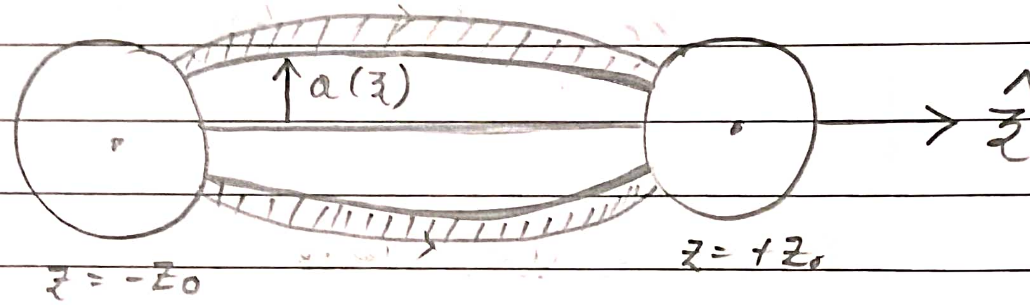
Let V be the volume of an axissymmetric flux tube with a flux ϕ.



$a(\phi, z) \rightarrow$ boundary of the flux tube with flux ϕ.

$$\phi = \int_0^a B_z(a,z) 2\pi a \, da$$

$$\Rightarrow \left.\frac{\partial \phi}{\partial a}\right|_z = 2\pi a B_z(a,z) \quad \text{—(1)}$$

Volume of flux tube $\rightarrow$

$$V = \pi \int_{-Z_0}^{+Z_0} a^2 dz$$, where $\pm Z_0$ are the ends of the flux tube.

$$\Rightarrow \frac{\partial V}{\partial \phi} = 2\pi \int_{-Z_0}^{+Z_0} a \left.\frac{\partial a}{\partial \phi}\right|_z dz$$

using (1) $\rightarrow$ $$\frac{\partial V}{\partial \phi} = 2\pi \int_{-Z_0}^{+Z_0} a \Big/ 2\pi a B_z(a,z) \, dz$$

$$\therefore \frac{\partial V}{\partial \phi} = \int_{-z_0}^{+z_0} \frac{dz}{B_z(a,z)}$$

We claim that once this $\frac{\partial V}{\partial \phi}$ or $\frac{\partial \phi}{\partial V}$ is determined by the initial condition, it will not vary by varying the magnets position.

This is because, the volume of flux tube containing flux ϕ can't change.

$V(\phi)$ is invariant

$\Rightarrow \frac{\partial V}{\partial \phi}$ is also invariant.

☆☆ To calculate the force, we can calculate the rate of work done by ext. agent in increasing the seperation.

This work done is stored as field energy and mutual interaction energy.

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We will estimate the stored field energy by integrating $\frac{B^2}{2\mu_0}$ over the entire volume.

Upon stretching by large L, the field strength along and near the axis would be very high and we can neglect the contribution of far-field in the total field energy.

We also approximate that after stretching $B_z(a,z)$ along a field line does not vary with z.

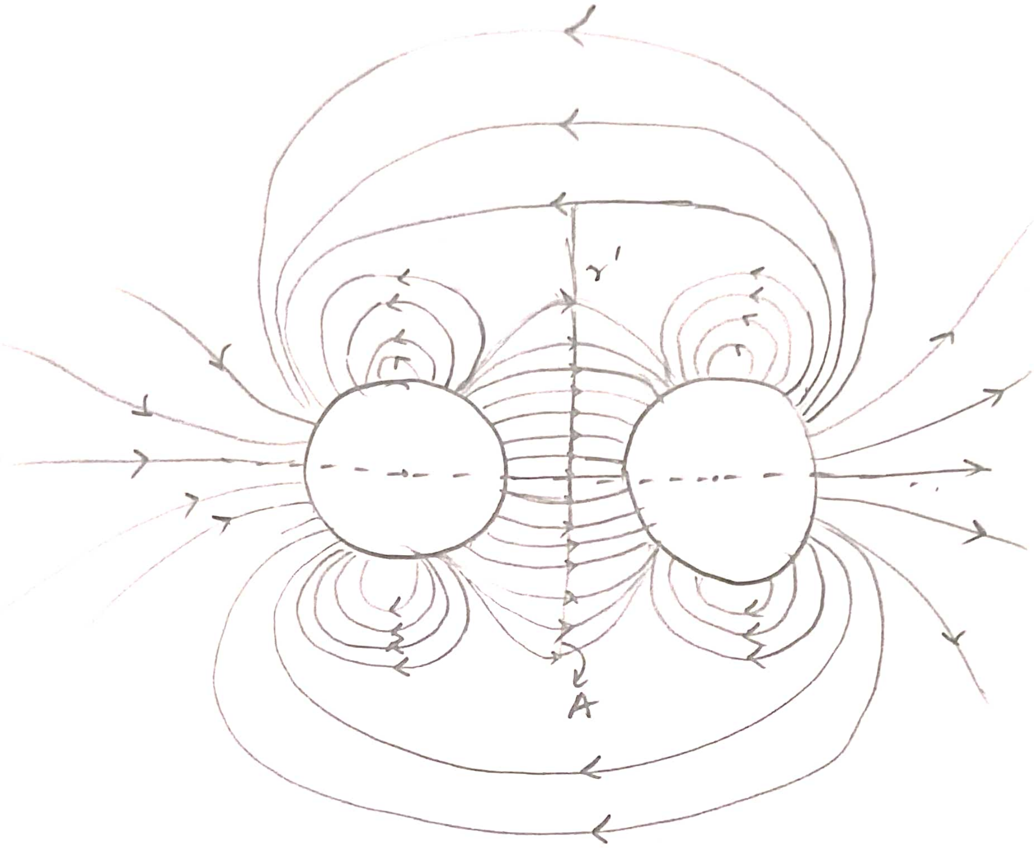
$$\therefore \int_{-L/2}^{+L/2} \frac{dz}{B_z(a,z)} \approx \frac{L}{B_z(a)} \approx \frac{dV}{d\phi}$$

or $$\boxed{B_z(a) \approx L \cdot \frac{d\phi}{dV}}$$

* We can neglect the radial contribution of the magnetic field.

$$B(a) = L\frac{d\phi}{dV} \quad \text{along a field line}$$

$r = a(z)$



SCHEMATIC DIAGRAM OF THE FIELD LINES IN INITIAL CONDITION.

r' is the point on $z=0$ plane, beyond which B_z changes sign.

ϕ till that point

$$\sim \frac{\mu_0 m}{R} \times C \qquad \left\{ C \equiv \text{dimensionless constant} \right\}$$

$$\frac{d\phi}{dV} \approx \frac{\phi}{V} \approx C' \frac{\mu_0 m}{R^4}.$$

field energy →

$$U = \frac{1}{2\mu_0}\int_V B^2(a)\, dV$$

We do the integral over the volume of all flux tubes, but neglect the ones which arent stretched axially.

$$\therefore \quad U = \frac{L^2}{2\mu_0}\int_V \left(\frac{d\phi}{dV}\right)^2 dV$$

The integral can be determined completely from the initial conditions.

Since we only want the dependency of U with L, m, R, we estimate

$$\int_V \left(d\phi/dV\right)^2 dV \approx C\,\frac{\mu_0^2 m^2}{R^8} \times R^3$$

$$\Rightarrow U = \frac{C \mu_0^2 L^2 m^2}{R^5}$$

$$|F| = \left| \frac{dU}{dL} \right| = \frac{c' \mu_0^2 L m^2}{R^5}$$

So upto a constant c' determined from initial condition. $\rightarrow$

$$\boxed{F = \frac{c' \mu_0^2 m^2}{R^5} L}$$