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Problem No 4

Let us denote the velocity obtained after acceleration with the proper acceleration g after the proper time τ as V. Let the Lorentz factor associated with it be denoted as $\gamma_0 = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$. Let us choose the reference frames A, B, C, D and four-velocities u, v, w, q such as stated in the third Hint. Then we can easly find some of the values of the four-velocities in some reference frames. Namely, by definition (we will write only three components, as the z-component is always zero):

$$u^A = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}, \; v^B = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}, \; w^C = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}, \; q^D = \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix}.$$

Additionally, using the previously introduced notation:

$$v^A = \begin{bmatrix} \gamma_0 c \\ \gamma_0 V \\ 0 \end{bmatrix}, \; w^B = \begin{bmatrix} \gamma_0 c \\ 0 \\ \gamma_0 V \end{bmatrix}.$$

By symmetry - or more carefully, by equating $u^A v^A = u^B v^B$ and $v^B w^B = v^C w^C$ - we easly obtain:

$$u^B = \begin{bmatrix} \gamma_0 c \\ -\gamma_0 V \\ 0 \end{bmatrix}, \; v^C = \begin{bmatrix} \gamma_0 c \\ 0 \\ -\gamma_0 V \end{bmatrix}.$$

This can also be obtained from the simple formula for relativistic velocity addition in 1 dimension. Such straightforward symmetry does not occur in the general case. We want to find the values of w^A and u^C. It can easily be done, using four-vector invariants. Namely:

$$w^A u^A = w^B u^B \implies w_t^A c = \gamma_0^2 c^2 \implies w_t^A = \gamma_0^2 c$$

$$w^A v^A = w^B v^B \implies \gamma_0^3 c^2 - \gamma_0 V w_x^A = \gamma_0 c^2 \implies w_x^A = \gamma_0^2 V$$

$$w^A w^A = w^C w^C \implies \gamma_0^4 c^2 - \gamma_0^4 V^2 - {w_y^A}^2 = c^2 \implies w_y^A = \gamma_0 V.$$

Ergo:

$$w^A = \begin{bmatrix} \gamma_0^2 c \\ \gamma_0^2 V \\ \gamma_0 V \end{bmatrix}.$$

Very similarly, through equating $u^C w^C = u^B w^B$, $u^B v^B = u^C v^C$ and $u^C u^C = u^A u^A$, we obtain:

$$u^C = \begin{bmatrix} \gamma_0^2 c \\ -\gamma_0 V \\ -\gamma_0^2 V \end{bmatrix}.$$

Let us note, that q^C can be calculated in the exactly the same way as w^A, although with switched signs. Therefore:

$$q^C = \begin{bmatrix} \gamma_0^2 c \\ -\gamma_0^2 V \\ -\gamma_0 V \end{bmatrix}.$$

Now, using the invariant $q^A u^A = q^C u^C$, we can obtain q_t^A:

$$q^A u^A = q^C u^C \implies q_t^A c = \gamma_0^4 c^2 - \gamma_0^3 V^2 - \gamma_0^3 V^2.$$

$$q_t^A = \frac{\gamma_0^3}{c}(\gamma_0 c^2 - 2V^2).$$

If the value of velocity after all the accelerations shall equal V, then clearly q_t^A must equal $\frac{c}{\sqrt{1-\frac{V^2}{c^2}}} = \gamma_0 c$. Therefore:

$$\frac{\gamma_0^3}{c}(\gamma_0 c^2 - 2V^2) = \gamma_0 c$$

$$\gamma_0 c^2 - 2V^2 = \frac{c^2}{\gamma_0^2}$$

$$\gamma_0 c^2 - 2V^2 = c^2 - V^2$$

$$\gamma_0 = 1 + \frac{V^2}{c^2}$$

$$1 = (1 + \frac{V^2}{c^2})\sqrt{1 - \frac{V^2}{c^2}}$$

$$1 = (1 + 2\frac{V^2}{c^2} + \frac{V^4}{c^4})(1 - \frac{V^2}{c^2})$$

$$0 = \frac{V^2}{c^2} - \frac{V^4}{c^4} - \frac{V^6}{c^6}$$

We can assume that $V \neq 0$:

$$0 = 1 - \frac{V^2}{c^2} - \frac{V^4}{c^4}.$$

Thats plain old quadratic equation with two solutions:

$$\frac{V^2}{c^2} = \frac{-1-\sqrt{5}}{2} \quad \text{or} \quad \frac{V^2}{c^2} = \frac{-1+\sqrt{5}}{2}$$

We can safely rule out the first solution, which gives us the final answer:

$$V = \sqrt{\frac{-1+\sqrt{5}}{2}}c$$

Note that the number under the square root is the reciprocal of the famous golden ratio, which is nice.