Physics Cup 2023

Problem 4 **Alptürk Eski**

Throughout the solution, I will call the Earth frame as rest frame. In the rest frame, the first phase of the motion $(0 \leq t < \tau)$ is simple. On the other hand, for $\tau < t$, in the rest frame, the motion is two-dimensional therefore, equations of motion are not simple. However, in a frame (S_1) whose relative velocity respect to the rest frame is given by $\overrightarrow{v_{10}} = v\hat{x}$, where v is the spaceship's speed at $t = \tau$ in the rest frame, the second phase of the motion $(\tau \leq t < 2\tau)$ is one dimensional. Because motions during the first and second phases are identical when observed from the rest frame and S_1, respectively, except 90° rotation, the final speed of the spaceship in S_1 is $\overrightarrow{v_1} = v\hat{y}$. Hereafter, S_1 will be useless and we will need to define a new frame according to the final speed of the spaceship as before. Define S_2 whose relative velocity respect to S_1 is $\overrightarrow{v_{21}} = v\hat{y}$. From the same considerations, I introduce the last frame as S_3 whose relative velocity respect to S_2 is $\overrightarrow{v_{32}} = -v\hat{x}$. In S_3 frame, the final velocity of the spaceship is $\overrightarrow{v_3} = -v\hat{y}$. Now, the problem is to obtain final velocity of the spaceship in the rest frame using velocity transformation equations throughout $S_3,\ S_2,\ S_1$.

Transformation from S_3 to S_2 frame:

$$v_{S_2y}^f = -\frac{v}{\gamma} \qquad v_{S_2x}^f = -v \qquad \vec{v}_{S_2}^f = -\frac{v}{\gamma}(\gamma\hat{x} + \hat{y})$$

Transformation from S_2 to S_1 frame:

$$v_{S_1y}^f = \frac{v_{S_2y}^f + v}{1 + \frac{vv_{S_2y}^f}{c^2}} = \frac{v - \frac{v}{\gamma}}{1 - \frac{v^2}{\gamma c^2}} = \frac{v(\gamma - 1)}{\gamma - \frac{v^2}{c^2}}$$

$$v_{S_1x}^f = \frac{v_{S_2x}^f}{\gamma\left(1 + \frac{vv_{S_2y}^f}{c^2}\right)} = \frac{-v}{\gamma\left(1 - \frac{v^2}{\gamma c^2}\right)} = -\frac{v}{\gamma - \frac{v^2}{c^2}}$$

$$\vec{v}_{S_1}^f = \frac{v}{\gamma - \frac{v^2}{c^2}}(-\hat{x} + (\gamma - 1)\hat{y})$$

Transformation from S_1 to the rest frame:

$$v_{fx} = \frac{v_{S_1x}^f + v}{1 + \frac{vv_{S_1x}^f}{c^2}} = \frac{v - \frac{v}{\gamma - \frac{v^2}{c^2}}}{1 - \frac{\frac{v^2}{c^2}}{\gamma - \frac{v^2}{c^2}}} = \frac{v\left(\gamma - \frac{v^2}{c^2}\right) - v}{\gamma - \frac{2v^2}{c^2}}$$

$$v_{fy} = \frac{v_{S_1y}^f}{\gamma\left(1 + \frac{vv_{S_1x}^f}{c^2}\right)} = \frac{\frac{v(\gamma - 1)}{\gamma - \frac{v^2}{c^2}}}{\gamma\left(1 - \frac{\frac{v^2}{c^2}}{\gamma - \frac{v^2}{c^2}}\right)} = \frac{v(\gamma - 1)}{\gamma\left(\gamma - \frac{2v^2}{c^2}\right)}$$

$$\overrightarrow{v_f} = \frac{v\left(\gamma - \frac{v^2}{c^2}\right) - v}{\gamma - \frac{2v^2}{c^2}}\hat{x} + \frac{v(\gamma - 1)}{\gamma\left(\gamma - \frac{2v^2}{c^2}\right)}\hat{y}$$

Final speed:

$$v_f^2 = \left[\frac{v\left(\gamma - \frac{v^2}{c^2}\right) - v}{\gamma - \frac{2v^2}{c^2}}\right]^2 + \left[\frac{v(\gamma - 1)}{\gamma\left(\gamma - \frac{2v^2}{c^2}\right)}\right]^2$$

From the problem text, we know this speed equals to v.

$$v^2 = \frac{\left[v\left(\gamma - \frac{v^2}{c^2}\right) - v\right]^2 + v^2\left(1 - \frac{1}{\gamma}\right)^2}{\left(\gamma - \frac{2v^2}{c^2}\right)^2}$$

Arrangement gives:

$$4\sqrt{1 - \frac{v^2}{c^2}} = 3 + \frac{v^2}{c^2} - 3\frac{v^4}{c^4}$$

Solving for v using a computer gives:

$$v = \sqrt{\frac{\sqrt{5} - 1}{2}}c \cong 0.786c$$