Physics Cup 2023
Problem 4: Spaceship

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# 1 Introduction

I have attempted to solve this problem by means of relative velocity and vectorial velocity addition in Special Relativity.

At the end of each acceleration step, let us consider an inertial frame co-moving with the rocket. In this frame, when the rocket starts accelerating in a different direction, the velocity of the rocket is parallel to the proper acceleration. This is because, initially, the rocket was at rest in this frame and experienced no further transverse acceleration from its path.

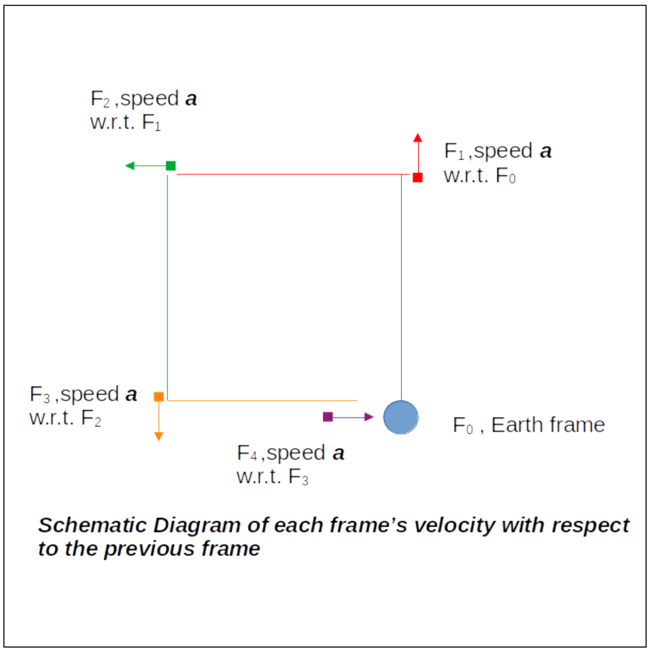



Figure 1: Note that this is **not** the trajectory of the rocket in Earth frame.

Let us label these frames $F_i$ at the end of $i^{th}$ acceleration step; also, let's label the Earth's frame as $F_0$. Let the speed of the rocket in $F_0$ frame after the first acceleration step be $v_1$. By symmetry, in the inertial frame, $F_i$, the rocket's speed after the next acceleration will also be $v_1$. (Since the rocket starts from rest and accelerates with an equal magnitude of proper acceleration for the same proper time $\tau$).

In our notation, let us denote the velocity of A in frame B as $\vec{v}_{A,B}$. The velocity of A in the C frame can be written as:

$$\vec{v}_{A,C} = \vec{v}_{A,B} \oplus \vec{v}_{B,C}$$

Where the $\oplus$ operator denotes velocity addition operation in SR. In $c = 1$ units, this is defined as:

$$\vec{v}_{A,B} \oplus \vec{v}_{B,C} = \frac{1}{1 + \vec{v}_{A,B} \cdot \vec{v}_{B,C}} \left[ \vec{v}_{B,C} + \alpha \vec{v}_{A,B} + (1 - \alpha) \frac{\vec{v}_{A,B} \cdot \vec{v}_{B,C}}{{v_{B,C}}^2} \vec{v}_{B,C} \right] \tag{1}$$

Here $\alpha$ is the inverse Lorentz factor $\sqrt{1 - {v_{B,C}}^2} = \frac{1}{\gamma}$.

We have been given that the speed of the rocket in $F_0$ after the $4^{th}$ acceleration step is equal to that after the $1^{st}$ acceleration. So we write:

$$\|\vec{v}_{F_4,F_0}\| = \|\vec{v}_{F_1,F_0}\| = v_1$$

Expanding the left-hand side in terms of the velocity addition formula,

$$\vec{v}_{F_4,F_0} = \vec{v}_{F_4,F_1} \oplus \vec{v}_{F_1,F_0}$$

Now,

$$\vec{v}_{F_4,F_1} = \vec{v}_{F_4,F_2} \oplus \vec{v}_{F_2,F_1}$$
$$\vec{v}_{F_4,F_2} = \vec{v}_{F_4,F_3} \oplus \vec{v}_{F_3,F_2}$$

Combining all of them we get the result,

$$\vec{v}_{F_4,F_0} = ((\vec{v}_{F_4,F_3} \oplus \vec{v}_{F_3,F_2}) \oplus \vec{v}_{F_2,F_1}) \oplus \vec{v}_{F_1,F_0} \tag{2}$$

# 2 Exploiting the same speed condition

We already know,

$$\|\vec{v}_{F_4,F_0}\| = \|\vec{v}_{F_1,F_0}\| = v_1 = a$$
$$\Rightarrow \|\vec{v}_{F_1,F_0}\| = \|\vec{v}_{F_4,F_1} \oplus \vec{v}_{F_1,F_0}\|$$

As per the shown sign convention, $\vec{v}_{F_1,F_0} = a\hat{x}$. Taking the velocity $\vec{v}_{F_4,F_1} = b\hat{x} + c\hat{y}$. Now we will use formula 1:

$$\vec{v}_{F_4,F_1} \oplus \vec{v}_{F_1,F_0} = \frac{1}{1 + ab} \left[\alpha_a (b\hat{x} + c\hat{y}) + a\hat{x} + (1 - \alpha_a) b\hat{x}\right]$$
$$\alpha_a = \sqrt{1 - a^2}$$
$$\Rightarrow \|\vec{v}_{F_4,F_1} \oplus \vec{v}_{F_1,F_0}\|^2 = \frac{1}{(1 + ab)^2} \left[(a + b)^2 + (1 - a^2)c^2\right] = \|\vec{v}_{F_1,F_0}\|^2 = a^2$$

Simplifying the last expression we get a relation between $a$, $b$ and $c$.

$$b^2(1 + a^2) + 2ab + c^2 = 0 \tag{3}$$

# 3 Evaluating $\vec{v}_{F_4,F_1}$

We have seen:

$$\vec{v}_{F_4,F_1} = (\vec{v}_{F_4,F_3} \oplus \vec{v}_{F_3,F_2}) \oplus \vec{v}_{F_2,F_1}$$

From the inertial frame $F_3$, the fourth acceleration step is towards the $-\hat{y}$ direction and since we have already argued that the magnitude will still be $a$, we know $\vec{v}_{F_4,F_3} = -a\hat{y}$. Similarly, $\vec{v}_{F_3,F_2} = -a\hat{x}$ and $\vec{v}_{F_2,F_1} = a\hat{y}$.

$$\Rightarrow \vec{v}_{F_4,F_3} \oplus \vec{v}_{F_3,F_2} = -a\hat{x} - a\sqrt{1 - a^2}\hat{y}$$

Therefore,

$$\vec{v}_{F_4,F_1} = \frac{1}{1 - a^2\sqrt{1 - a^2}} \left[a(1 - \sqrt{1 - a^2})\hat{y} - a\sqrt{1 - a^2}\hat{x}\right] \tag{4}$$

# 4 Finding the required speed $a$

Noting that in our equation 3,

$$b = \frac{-a\sqrt{1-a^2}}{1-a^2\sqrt{1-a^2}}$$

$$c = \frac{a(1-\sqrt{1-a^2})}{1-a^2\sqrt{1-a^2}}$$

For sake of brevity, let's substitute $X = \sqrt{1-a^2}$ and $Y = 1 - a^2\sqrt{1-a^2} = 1 - a^2X$. Now plugging $b$, $c$ in equation 3,

$$\frac{a^2(1+a)^2X^2}{Y^2} + \frac{a^2(1-X)^2}{Y^2} = \frac{2a^2X}{Y}$$

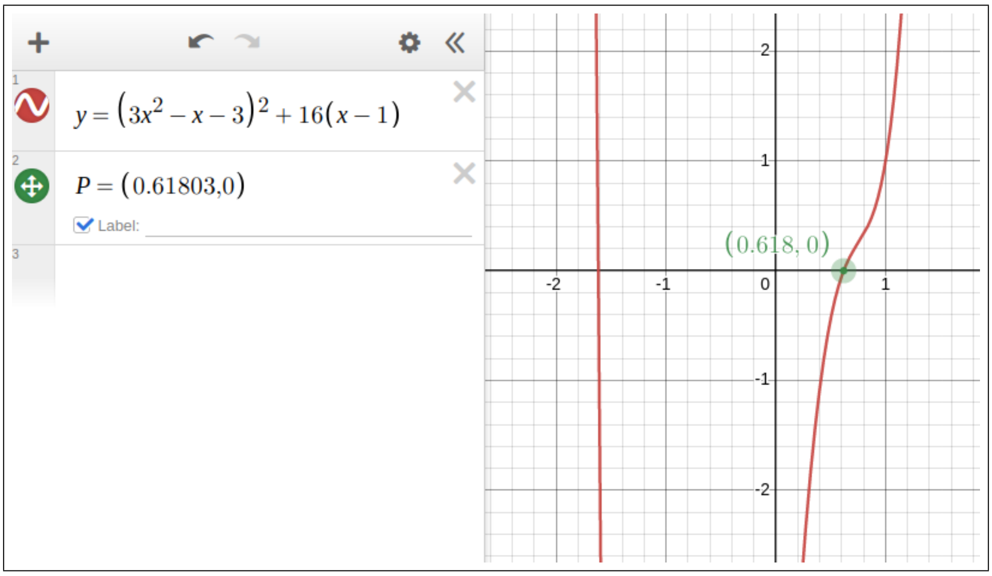
$$\Rightarrow (1-X)^2 + X^2(1+a^2) = 2XY = 2X(1-a^2X)$$

$$(2+3a^2)X^2 - 4X + 1 = 0$$

Substituting $X$ back as $\sqrt{(1-a^2)}$ and $w = a^2$:

$$(4X)^2 = 16(1-w) = [(1-u)(2+3u)+1]^2$$

$$\Rightarrow (3w^2 - w - 3)^2 + 16(w-1) = 0$$




The positive root of this polynomial (which has to be $< 1$ to obey SR) is the required value of $a^2$. I have plotted the graph to find the root to be:

$$w = a^2 = 0.618$$

Therefore the speed of the rocket at the end of $t = \tau$ is **0.786 c**. Where c is speed of light.