Physics Cup 2023, Problem 4

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If we are observing an object moving with constant proper acceleration of magnitude g which is initially at rest, we observe it to obtain some velocity v as proper time τ passes in the rest frame of the object. Since this is simple (1+1)D accelerated motion, this velocity is in the same direction as the acceleration.
The direction of proper acceleration is in the x direction as observed from the rest frame of the spaceship during the first interval of duration τ. Let's call the Earth frame S_0. During the first interval of acceleration the spaceship obtains a velocity $\vec{v}(\tau)$, whose magnitude is v, in the S_0 frame.
At $t = \tau$ the acceleration changes direction to point in the y direction in the spaceship frame. From the frame S_1, in which the spaceship is at rest in the instant the acceleration changes direction, the motion proceeds analogously to the motion during the first interval in the S_0 frame. Before the acceleration direction changes again the spaceship attains velocity of magnitude v in S_1 but in the y_1 direction, the y_1 axis of the S_1 frame corresponding to the y axis of the spaceship frame at the end of the first interval when the two systems had zero relative velocity and when we pick them to coincide. The S_1 frame is moving with velocity v with respect to the S_0 system in its x_0 direction, the axis along which the motion was taking place during the first interval.
Similarly we pick the S_2 frame as the one which moves in the y_0 direction with velocity v with respect to S_1 and in which the spaceship is at rest at the end of the second interval of acceleration. The S_3 frame shall move in the $-x_2$ direction with speed v with respect to S_2 and S_4 shall move in the $-y_3$ direction with the same speed with respect to S_3, where we assign to the S_i system the axes x_i and y_i $(i = 0, 1, 2, 3, 4)$ such that in the instant the spaceship is at rest in S_i (at the end of the i-th interval) the spaceship frame and S_i coincide.
The velocity of the spaceship at $t = \tau$ in the S_1 frame is zero by construction as well as at $t = 4\tau$ in S_4. What remains is to find the corresponding velocities in S_0, or, in other words, to apply the relevant Lorentz boosts which connect the systems to find how the velocity transforms.
Lorentz boosts can be interpreted as rotations in Minkowski spacetime in which the time component of some generic position 4-vector $\vec{r} = (ict, x, y, z)$ has an added factor of the imaginary unit. Thus a Lorentz boost in the x direction is represented as a rotation of the 4-vector in the ict-x plane by an imaginary angle α, where the relations between the relativistic parameters γ and β and α are

$$\cos(\alpha) = \gamma \tag{1}$$
$$\sin(\alpha) = i\gamma\beta \tag{2}$$

In matrix form the Lorentz boost in the x direction is then given by (where we ignore the third spacial component because motion is confined to 2+1 dimensions)

$$R_x(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

(where we ignore the third spacial component because motion is confined to 2+1 dimensions). The boost in the y direction is, analogously, a rotation in the ict-y plane.

$$R_y(\alpha) \begin{pmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \tag{4}$$

In order to find the 3-velocity transformations we first introduce the 4-velocity which is defined as

$$\vec{\xi} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}\tau} \tag{5}$$

and which transforms like any 4-vector. Because $\mathrm{d}t = \gamma\,\mathrm{d}\tau$, it is easy to see that $\vec{\xi} = \gamma(ic, v_x, v_y, v_z)$ (where we will henceforth drop the third spacial component because it is always zero).

So, the proper velocity of the spaceship at $t = \tau$ in S_1 is $\vec{\xi_1}(\tau) = (ic, 0, 0)$ and at $t = 4\tau$ in S_4 is $\vec{\xi_4}(4\tau) = (ic, 0, 0)$. By construction, the boost $R_x(\alpha)$ transforms 4-vectors in S_0 to 4-vectors in S_1, $R_x(-\alpha)$ is the inverse and α is the angle of rotation corresponding to a boost of speed v. Similarly, $R_y(-\alpha)$ brings vectors from S_2 to S_1, $R_x(\alpha)$ from S_3 to S_2 and $R_y(\alpha)$ form S_4 to S_3. Therefore, the 4-velocity as seen from S_0 at times $t = \tau$ and $t = 4\tau$ is

$$\vec{\xi_0}(\tau) = R_x(-\alpha)\vec{\xi_1}(\tau) \tag{6}$$

$$\vec{\xi_0}(4\tau) = R_x(-\alpha)R_y(-\alpha)R_x(\alpha)R_y(\alpha)\vec{\xi_4}(4\tau) \tag{7}$$

Once the matrix multiplication is carried out, one gets

$$\vec{\xi_0}(\tau) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ic \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

$$\vec{\xi_0}(4\tau) = \begin{pmatrix} \cos(\alpha)^4 - 2\cos(\alpha)^3 + 2\cos(\alpha) & ... & .., \\ -\left(\cos(\alpha)^3 - 2\cos(\alpha)^2 + 1\right)\sin(\alpha) & ... & ... \\ -\left(\cos(\alpha)^2 - \cos(\alpha)\right)\sin(\alpha) & ... & ... \end{pmatrix} \begin{pmatrix} ic \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

where we have disregarded the second and third columns of the second matrix because they multiply with zero to give the relevant vector. Moreover, we can even disregard any matrix elements other than the first element of the first row in both of the matrices because we are interested in the magnitudes of the corresponding 3-vectors which are completely "encoded" in the gamma factors, and the time component of the 4-vector is precisely γic. Therefore, the time components of $\vec{\xi_0}(4\tau)$ and $\vec{\xi_0}(\tau)$ have to be equal as their corresponding speeds are given to be equal, which means their gamma factors are as well, so the elements in the first row and column of the above matrices have to be equal:

$$\Rightarrow \cos(\alpha)^4 - 2\cos(\alpha)^3 + 2\cos(\alpha) = \cos(\alpha) \tag{10}$$

$$\Rightarrow \gamma^4 - 2\gamma^3 + 2\gamma = \gamma \tag{11}$$

$$\Leftrightarrow \gamma(\gamma - 1)(\gamma^2 - \gamma - 1) = 0 \tag{12}$$

$$\Rightarrow \gamma \in \left\{0, 1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right\} \tag{13}$$

where in eq.10 the relation eq.1 was used. Since $\gamma \geq 1$, the first and fourth solution are impossible and $\gamma = 1$ is the trivial solution which corresponds to zero acceleration.

$$\gamma = \varphi = \frac{1+\sqrt{5}}{2} \tag{14}$$

$$\Rightarrow v = c\varphi^{-1/2} \tag{15}$$