Problem 4 – Spaceship

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1 Task

A spaceship takes off from the Earth at $t = 0$, and henceforth keeps the modulus of its proper acceleration equal to g; here and in what follows, t denotes the spaceship's proper time. Thus, the astronauts on board will always feel a constant free fall acceleration g. However, the direction of the proper acceleration is changed four times at equal intervals, by turning the engines counterclockwise by 90°. So, the proper acceleration is:

- parallel to the x-axis when $0 \leq t < \tau$
- parallel to the y-axis when $\tau \leq t < 2\tau$
- antiparallel to the x-axis when $2\tau \leq t < 3\tau$
- antiparallel to the y-axis when $3\tau \leq t < 4\tau$

It turns out that the spaceship's speed relative to the Earth takes exactly the same value v at $t = \tau$ and $t = 4\tau$. Find this value v!

2 Lorentz transformation of 4-vectors

Let a be a contravariant 4-vector in the inertial system S, for example:

- 4-position: $X = (ct, x, y, z)^T$
- 4-velocity: $V = \gamma \cdot \frac{\mathrm{d}X}{\mathrm{d}t} = \gamma \cdot (c, \vec{v})^T$, where $\vec{v} = (\dot{x}, \dot{y}, \dot{z})$ is the velocity observed in S and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$
- 4-acceleration: $B = \gamma \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \gamma \cdot \frac{\mathrm{d}}{\mathrm{d}t} (\gamma c, \gamma \vec{v})^T$

In the inertial system S' moving relative to S with velocity u in the x-direction

$$a' = \Lambda \cdot a,$$

where

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here, $\beta = \frac{u}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Furthermore, $a = \Lambda^{-1} \cdot a'$, where

$$\Lambda^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3 Solution

Let the velocity of the spaceship relative to the Earth at time τ be v. In an inertial frame S_1 moving relative to the Earth with velocity v parallel to the x-axis, the spaceship appears to accelerate from rest in the y-direction from then on. To an observer in S_1, the process looks the same as it did to an observer on Earth when the spaceship accelerated in the x-direction. At the time 2τ, the spaceship thus reaches the velocity v in S_1. Let further S_2 be the inertial frame moving in the y-direction relative to S_1 with velocity v and S_3 be the inertial frame traveling antiparallel to the x-axis relative to S_2 with velocity v. In S_3 the 4-velocity of the spaceship at time 4τ is

$$V_{S_3} = c\gamma \cdot (1, 0, -\beta, 0)^T \,.$$

Here, $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The 4-velocity measured by an observer on Earth can be found by the Lorentz transformations from S_3 to S_2, S_2 to S_1 and S_1 to the Earth's inertial frame:

$$V_{\text{Earth}} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & 0 & \beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot V_{S_3} = c\gamma \cdot \begin{pmatrix} \gamma^3 - 2\beta^2\gamma^2 \\ \beta\gamma^3 - \beta^3\gamma^2 - \beta\gamma^2 \\ \beta\gamma^2 - \beta\gamma \\ 0 \end{pmatrix}$$

If the spaceship is moving with velocity v relative to the Earth at this time, then the first component of the 4-velocity is $c\gamma$, i.e. $\gamma^3 - 2\beta^2\gamma^2 = 1$. Hence,

$$\begin{aligned} 1 - \beta^2 = \gamma^{-2} &= \gamma - 2\beta^2 \\ \frac{1}{\sqrt{1-\beta^2}} &= \gamma = 1 + \beta^2 \\ \frac{1}{1-\beta^2} &= 1 + 2\beta^2 + \beta^4 \\ 1 &= 1 + 2\beta^2 + \beta^4 - \beta^2 - 2\beta^4 - \beta^6 \\ 0 &= \beta^2 \cdot \left(1 - \beta^2 - \beta^4\right) \end{aligned}$$

We are looking for a $\beta > 0$ that satisfies the upper equation:

$$\begin{aligned} \beta^2 &= -\frac{1}{2} + \sqrt{\frac{1}{4} + 1} \\ \beta &= \sqrt{\sqrt{1.25} - 0.5} \end{aligned}$$

Thus, the spaceship moves with velocity $v = \sqrt{\sqrt{1.25} - 0.5} \cdot c \approx 0.78615c$ relative to the Earth at times τ and 4τ.

4 Appendix: Finding τ

We consider the time period $0 \leq t < \tau$. Let t_E be the time that has passed on earth. The 4-acceleration in the spaceship's inertial frame is $B' = (0, g, 0, 0)^T$, so for an observer on Earth

$$B = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot B' = \begin{pmatrix} \beta\gamma g \\ \gamma g \\ 0 \\ 0 \end{pmatrix}.$$

Thus, we obtain in particular

$$g = \frac{\mathrm{d}}{\mathrm{d}t_\mathrm{E}} (\gamma v) .$$

Integration and subsequent transformations lead to

$$v(t_\mathrm{E}) = \frac{g t_\mathrm{E}}{\sqrt{1 + \left(\frac{g t_\mathrm{E}}{c}\right)^2}},$$

taking into account the initial condition $v(0) = 0$. In the spaceship, time

$$t = \int_0^{t_\mathrm{E}} \frac{\mathrm{d}t'}{\gamma(t')} = \int_0^{t_\mathrm{E}} \sqrt{1 - \frac{v(t')^2}{c^2}} \mathrm{d}t' = \int_0^{t_\mathrm{E}} \frac{\mathrm{d}t'}{\sqrt{1 + \left(\frac{g t'}{c}\right)^2}} = \frac{c}{g} \cdot \operatorname{arsinh}\left(\frac{g t_\mathrm{E}}{c}\right)$$

has passed. Converting to t_E yields

$$t_\mathrm{E} = \frac{c}{g} \cdot \sinh\left(\frac{gt}{c}\right)$$

and inserting gives

$$v(t) = \frac{c \cdot \sinh\left(\frac{gt}{c}\right)}{\sqrt{1 + \sinh^2\left(\frac{gt}{c}\right)}} = c \cdot \tanh\left(\frac{gt}{c}\right).$$

Therefore,

$$v(\tau) = c \cdot \tanh\left(\frac{g\tau}{c}\right).$$

Thus, we can specify the time that has elapsed in the spaceship when the direction of acceleration changes:

$$\tau = \frac{c}{g} \cdot \operatorname{artanh}\left(\frac{v(\tau)}{c}\right) = \frac{c}{g} \cdot \operatorname{artanh}(0.78615) = 1.06127 \cdot \frac{c}{g}$$

This gives $\tau = 375.37$ days using the well-known constants.