Physics Cup 2023 - Problem 4

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1 The effect of each impulse

Let us consider the rest frame of the spaceship at the start of each period of constant acceleration. The spaceship will accelerate for a proper time τ in some direction. This will lead to the spaceship attaining a rectilinear motion with a speed of v. Since τ is the same for each period of acceleration, the speed v will also be the same, as the physical situation is identical (that of the spaceship leaving from rest and accelerating at proper acceleration g for proper time τ), and hence will also be equal to the speed v mentioned in the problem text, which is the speed at the end of the first period of acceleration.

Hence, each of the four periods of constant acceleration amounts to a Lorentz boost with speed v in a different direction. Let us consider a transformation Λ that includes a boost with speed v and a rotation with angle 90°:

$$\Lambda = R(90^\circ)B(v), \tag{1}$$

where the boost is along the x-axis of the initial frame. As long as the x-axis of the initial frame is aligned with the direction of the acceleration of the spaceship, Λ amounts to a transformation from the rest frame of the ship at the beginning of an acceleration period to that at the end of the period, and assures that the x-axis of the new frame is aligned to the new direction of acceleration of the ship. Hence, applying Λ successively four times will transform the initial rest frame of the spaceship into the rest frame of the spaceship after all four periods of acceleration.

2 Expressing Λ and calculating v

Note: In order to simplify expressions, I will only write down the first three components of four-vectors. It is understood that the z-component of all four-vectors is 0.

The rotation and boost matrices can be written as

$$R(90^\circ) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 90^\circ & \sin 90^\circ \\ 0 & -\sin 90^\circ & \cos 90^\circ \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}; \tag{2}$$

$$B(v) = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3}$$

where $\beta = \frac{v}{c}$ and $\gamma = (1-\beta^2)^{-1/2}$. Hence,

$$\Lambda = R(90^\circ)B(v) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ 0 & 0 & 1 \\ \beta\gamma & -\gamma & 0 \end{pmatrix}. \tag{4}$$

Let us consider the four-vector $\mathbf{X} = (1,\ 0,\ 0)$ (i.e. the interval between two events that take place at the same place in the Earth frame a unit time apart). Due to the motion of the spaceship, the time component of this four-vector will be dilated by a factor related to the relative velocity of the Earth with respect to the ship. According to the standard time-dilation result, the factor of dilation is the Lorentz factor of said speed, which is also the speed of the ship with respect to Earth. Hence, by computing the transformation of this four-vector into the frame of the ship after all four impulses and considering its time component, we can find the Lorentz factor of the ship at the end of the accelerated motion.

Let us now compute the transformation of this four-vector. We will have:

$$\Lambda\mathbf{X} = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ 0 & 0 & 1 \\ \beta\gamma & -\gamma & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \\ \beta\gamma \end{pmatrix};$$

$$\Lambda^2\mathbf{X} = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ 0 & 0 & 1 \\ \beta\gamma & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ 0 \\ \beta\gamma \end{pmatrix} = \begin{pmatrix} \gamma^2 \\ \beta\gamma \\ \beta\gamma^2 \end{pmatrix};$$

$$\Lambda^3\mathbf{X} = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ 0 & 0 & 1 \\ \beta\gamma & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \gamma^2 \\ \beta\gamma \\ \beta\gamma^2 \end{pmatrix} = \begin{pmatrix} \gamma^3 - \beta^2\gamma^2 \\ \beta\gamma^2 \\ \beta\gamma^3 - \beta\gamma^2 \end{pmatrix};$$

$$\Lambda^4\mathbf{X} = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ 0 & 0 & 1 \\ \beta\gamma & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \gamma^3 - \beta^2\gamma^2 \\ \beta\gamma^2 \\ \beta\gamma^3 - \beta\gamma^2 \end{pmatrix} = \begin{pmatrix} \gamma^4 - 2\beta^2\gamma^3 \\ \beta\gamma^3 - \beta\gamma^2 \\ \beta\gamma^4 - \beta\gamma^3 - \beta^3\gamma^3 \end{pmatrix}.$$

Hence, the Lorentz factor of the final speed of the ship is

$$\begin{aligned} \gamma_4 &= \gamma^4 - 2\beta^2\gamma^3 \\ &= \gamma^4 - 2\gamma(\gamma^2 - 1). \end{aligned} \tag{5}$$

In order for this final speed to be equal to v, its Lorentz factor must be equal to γ:

$$\begin{aligned} &\gamma_4 = \gamma \\ \Longleftrightarrow\ &\gamma^4 - 2\gamma^3 + 2\gamma = \gamma \\ \Longleftrightarrow\ &\gamma^3 - 2\gamma^2 + 1 = 0 \\ \Longleftrightarrow\ &(\gamma - 1)(\gamma^2 - \gamma - 1) = 0. \end{aligned} \tag{6}$$

The first solution of Eq. (6) is the trivial $\gamma = 1$, in which case the ship doesn't move at all, and its speed is always 0. The other solution corresponds to

$$\gamma^2 - \gamma - 1 = 0 \Longrightarrow \gamma = \frac{1+\sqrt{5}}{2}, \tag{7}$$

with the other solution being negative and hence non-physical. This value of γ leads to

$$v = \sqrt{1 - \frac{1}{\gamma^2}} c \Longrightarrow v = \sqrt{\frac{\sqrt{5}-1}{2}} c \approx 0.78615c. \tag{8}$$