Physics Cup 2023 Problem 4

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Contents

1 Introduction

First, I will introduce the general idea of the solution and thereafter give the respective solution with the algebraic manipulation left out. This is done as this particular solution (for an equation) is algebraically pretty heavy (i.e. not an ideal or a beautiful solution) but as there are some niceties in that manipulation (and for double-checking) I decided to write it out in the extra chapters. Similarly, out of my own interest, the respective proper time, τ, for which this situation is possible is calculated as well.

As we will soon see, this solution heavily uses relativistic velocity addition. Hence, to make sure everything checks out, these are proven using Lorentz transformation in the extra chapters.

2 Layout of the idea

We start working in the frame of reference of the earth, S. We define (for the rest of the problem) the positive x-axis to point in the direction of the spaceship's proper acceleration during $0 \leq t < \tau$ and positive y-axis to point in the direction of the spaceship's proper acceleration during $\tau \leq t < 2\tau$.

Now, in the frame of S, the spaceship starts from rest and accelerates, reaching the speed v. Now, we can go to the inertial frame of reference, S', for which the spaceship is again momentarily at rest. I.e. the velocity of S' with respect to S is $(u'_x, u'_y,) = (v, 0)$ However, now the situation is identical (but the direction of proper acceleration is different), so the spaceship reaches the velocity $(v'_x, v'_y) = (0, v)$ in the frame of S'.

We can repeat the same process of changing to the new reference frame of the spaceship after a certain burst of acceleration until we are at $t = 4\tau$. At this point we have the spaceship moving at $(v'''_x, v'''_y) = (0, -v)$ in the reference frame S''' moving at $(u'''_x, u'''_y) = (-v, 0)$ with respect to S'' moving at $(u''_x, u''_y) = (0, v)$ with respect to S'.

Having this in mind, we can just repeatedly apply the velocity addition formulae to find the velocity of the spaceship in $S'' \to S' \to S$. Finally, we can equate the respective speed with v and solve for v.

The relativistic velocity formulae work as follows. If a frame K' moves with $(v_i, v_j) = (v, 0)$ with respect to K and a particle has a velocity of (u'_i, u'_j) (note that axes i and j

are perpendicular), then the velocity of the particle in $K, (u_i, u_j)$ is found as follows:

$$u_i = \frac{u_i' + v}{1 + u_i' v/c^2}$$
$$u_j = \frac{u_j'}{\gamma(1 + u_i' v/c^2)},$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

3 Solution

Let

$$\beta = \frac{v}{c} > 0.$$

We note that for each switch in frame we do, the Lorentz factor stays the same as the relative speed between the frames is always v. Hence

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}.$$

Now, the velocity of the spaceship in S''' is $(v_x''', v_y''') = (0, -v)$ and hence the velocity in $S'', (v_x'', v_y'')$ is ($\vec{u}'''$ in S'' is parallel to the x-axis so $i = x$ and $j = y$):

$$v_x'' = -v$$
$$v_y'' = \frac{-v}{\gamma}.$$

The velocity of the spaceship in $S', (v_x', v_y')$ is ($\vec{u}''$ in S' is parallel to the y-axis so $i = y$ and $j = x$):

$$v_x' = \frac{-v}{\gamma\left(1 + \frac{\beta^2}{\gamma}\right)}$$
$$v_y' = \frac{-\frac{v}{\gamma} + v}{1 - \frac{\beta^2}{\gamma}}.$$

The velocity of the spaceship in S, (v_x, v_y) is ($\vec{u}'$ in S is parallel to the x-axis so $i = x$ and $j = y$):

$$
\begin{aligned}
v_x &= \frac{\frac{-v}{\gamma\left(1-\frac{\beta^2}{\gamma}\right)} + v}{1 - \frac{\beta^2}{\gamma\left(1-\frac{\beta^2}{\gamma}\right)}} \\
v_y &= \frac{\frac{-\frac{v}{\gamma}+v}{1-\frac{\beta^2}{\gamma}}}{\gamma\left(1 - \frac{\beta^2}{\gamma\left(1-\frac{\beta^2}{\gamma}\right)}\right)}.
\end{aligned}
$$

Dividing both sides by v and simplifying:

$$
\begin{aligned}
\frac{v_x}{v} &= \frac{\gamma - \beta^2 - 1}{\gamma - 2\beta^2} \\
\frac{v_y}{v} &= \frac{\gamma - 1}{\gamma^2 - 2\gamma\beta^2}
\end{aligned}
$$

We want to find the v for which the speed of the spaceship in S is the same at $t = \tau$ and $t = 4\tau$. Hence, we get the equation:

$$
\begin{aligned}
& & v_x^2 + v_y^2 &= v^2 \\
\implies & & \left(\frac{v_x}{v}\right)^2 + \left(\frac{v_x}{v}\right)^2 &= 1 \\
\iff & \left(\frac{\gamma - \beta^2 - 1}{\gamma - 2\beta^2}\right)^2 + & \left(\frac{\gamma - 1}{\gamma^2 - 2\gamma\beta^2}\right)^2 &= 1.
\end{aligned}
$$

Plugging in $\gamma = 1/\sqrt{1-\beta^2}$ and solving for β (remembering that $\beta > 0$), we get that

$$
\begin{aligned}
\beta &= \sqrt{\frac{\sqrt{5}-1}{2}} \\
\iff & \boxed{v = c\sqrt{\frac{\sqrt{5}-1}{2}}.}
\end{aligned}
$$

As a side note, the Lorentz factor is actually equal to the golden ratio (see 4.1) $\gamma = \varphi = (1+\sqrt{5})/2$ and thus $v = c\sqrt{1 - 1/\varphi^2}$.

4 Extra stuff

4.1 Solving the equation

We have the equation

$$\left(\frac{\gamma-\beta^2-1}{\gamma-2\beta^2}\right)^2+\left(\frac{\gamma-1}{\gamma^2-2\gamma\beta^2}\right)^2=1$$

to solve for β. It is actually easier to solve for γ first and then β as we will see soon. Hence, we will use the following relation:

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\beta^2}}\\ \implies \beta^2 &= 1-\frac{1}{\gamma^2}.\end{aligned}$$

Now, back to the equation at hand. First we will rewrite it a little:

$$\begin{aligned}&\left(\frac{\gamma-\beta^2-1}{\gamma-2\beta^2}\right)^2+\left(\frac{\gamma-1}{\gamma^2-2\gamma\beta^2}\right)^2=1\\ \implies\quad &(\gamma-\beta^2-1)^2+\left(\frac{\gamma-1}{\gamma}\right)^2=(\gamma-2\beta^2)^2\\ \iff\quad &(\gamma-2\beta^2)^2-(\gamma-\beta^2-1)^2=\left(\frac{\gamma-1}{\gamma}\right)^2\\ \iff\quad &(1-\beta^2)(2\gamma-3\beta^2-1)=\left(\frac{\gamma-1}{\gamma}\right)^2.\end{aligned}$$

Now, when we plug in our expression for β with respect to γ, we get that

$$\begin{aligned}&\frac{1}{\gamma^2}\left(2\gamma-3\left(1-\frac{1}{\gamma^2}\right)-1\right)=\left(\frac{\gamma-1}{\gamma}\right)^2\\ \implies\quad &2\gamma+\frac{3}{\gamma^2}-4=\gamma^2-2\gamma+1\\ \iff\quad &\gamma^2-4\gamma+5-\frac{3}{\gamma^2}=0\\ \implies\quad &\gamma^4-4\gamma^3+5\gamma^2-3=0\\ \iff\quad &(\gamma^2-\gamma-1)(\gamma^2-3x+3)=0.\end{aligned}$$

The second quadratic equation has a negative discriminant so it has no real roots. However, the first quadratic equation has the roots:

$$\frac{1+\sqrt{5}}{2} \text{ and } \frac{1-\sqrt{5}}{2}.$$

The second root is negative which is non-physical for the Lorentz factor so:

$$\gamma = \frac{1+\sqrt{5}}{2} = \varphi.$$

I.e. the Lorentz factor is equal to the golden ratio.

4.2 Calculation of τ

We can find τ from v as we know that with proper acceleration g the spaceship achieves speed v in proper time τ. We note that during the initial acceleration the acceleration and velocity are parallel.

Let the speed of the spaceship with respect to proper time t be $v(t)$. There always exists a frame K for which the spaceship is instantaneously at rest. In this frame after $\mathrm{d}t$ the spaceship reaches a speed of $g\ \mathrm{d}t$. Now, using the velocity addition formula (both the acceleration and velocity of the spaceship are along i):

$$v(t+\mathrm{d}t) = \frac{v(t) + g\ \mathrm{d}t}{1 + v(t)g\ \mathrm{d}t/c^2}.$$

The right hand sides simplifies to (neglecting $\mathcal{O}(\mathrm{d}t^2)$ terms):

$$\frac{v(t) + g\ \mathrm{d}t}{1 + v(t)g\ \mathrm{d}t/c^2} \cong v(t) + g\left(1 - \frac{v(t)^2}{c^2}\right)\mathrm{d}t.$$

Thus we get the separable differential equation:

$$\mathrm{d}v = g\left(1 - \frac{v^2}{c^2}\right)\mathrm{d}t$$
$$\implies \frac{\mathrm{d}v}{1 - v^2/c^2} = g\ \mathrm{d}t.$$

Thus:

$$\begin{aligned}
\int_0^v \frac{\mathrm{d}v}{1-v^2/c^2} &= \int_0^\tau g\ \mathrm{d}t. \\
\iff c\operatorname{arctanh}\left(\frac{v}{c}\right) &= g\tau \\
\iff \qquad\qquad \tau &= \frac{c}{g}\operatorname{arctanh}\left(\frac{v}{c}\right) \\
&= \frac{c}{g}\operatorname{arctanh}\left(\sqrt{1-\frac{1}{\varphi^2}}\right) \\
&\approx \frac{1{,}06c}{g}.
\end{aligned}$$

4.3 Proof of velocity addition formulae

Let S' be a frame moving along the i-axis with a velocity of v with respect to the frame S. A particle has the velocity (u_i', u_j') in S'. I.e. we know that

$$\begin{aligned}
\frac{q_i'}{t'} &= u_i' \\
\frac{q_j'}{t'} &= u_j'.
\end{aligned}$$

Here, q_k is displacement along axis k.

We can use Lorentz transformations to find $q_i, q_j,$ and t,

$$\begin{aligned}
q_i &= \gamma(q_i' + vt') \\
q_j &= q_j' \\
t &= \gamma\left(t' + \frac{vq_i'}{c^2}\right),
\end{aligned}$$

where γ is the Lorentz factor $\gamma = 1/\sqrt{1-v^2/c^2}$.

Now,

$$\begin{aligned}
u_i &= \frac{q_i}{t} \\
&= \frac{\gamma(q_i' + vt')}{\gamma(t' + vq_i'/c^2)} \\
&= \frac{q_i' + vt'}{t' + vq_i'/c^2} \\
&= \frac{q_i'/t' + v}{1 + (q_i'/t')v/c^2} \\
&= \frac{u_i' + v}{1 + u_i'v/c^2},
\end{aligned}$$

and

$$\begin{aligned}
u_j &= \frac{q_j}{t} \\
&= \frac{q_j'}{\gamma(t' + vq_i'/c^2)} \\
&= \frac{q_j'/t'}{\gamma(1 + (q_i'/t')v/c^2)} \\
&= \frac{u_j'}{\gamma(1 + u_i'v/c^2)}.
\end{aligned}$$