Problem No 4 - Gil Ronen

1 Velocity addition

If a body is moving inside a 2D plane with a velocity $\boldsymbol{v} = (v_x, v_y)$, It's velocity in a moving frame of reference moving with a velocity $v\hat{\boldsymbol{x}}$ is:

$$\Lambda_{v\hat{\boldsymbol{x}}}(v_x, v_y) = \left(\frac{v_x - v}{1 - vv_x}, \frac{v_y}{1 - vv_x}\sqrt{1 - v^2}\right)$$

Where $\Lambda_{v\hat{\boldsymbol{x}}}$ denotes the transformation under the change of reference frame which moves with velocity $v\hat{\boldsymbol{x}}$ ("Boost"). This is a known basic result of special relativity. An analogy exists for transformations in the y axis:

$$\Lambda_{v\hat{\boldsymbol{y}}}(v_x, v_y) = \left(\frac{v_x}{1 - vv_y}\sqrt{1 - v^2}, \frac{v_y - v}{1 - vv_y}\right)$$

2 The spaceship

If the spaceship experiences a constant proper acceleration g for a proper time τ. It will end up with a velocity v. We could work out the explicit expression $v(g, \tau)$ but it is not necessary at this point.

2.1 $t = \tau$

We can calculate the velocity at $t = \tau$ by boosting the velocity vector by $-v\hat{\boldsymbol{x}}$.

$$\boldsymbol{v}(\tau) = \Lambda_{-v\hat{\boldsymbol{x}}}(0, 0) = (v, 0)$$

2.2 $t = 2\tau$

In the boosted $\Lambda_{-v\hat{\boldsymbol{x}}}(\cdot)$ frame, the spaceship starts from rest and then aquires a velocity $(0, v)$. Therefore at $t = 2\tau$ the velocity w.r. to the earth is:

$$\boldsymbol{v}(2\tau) = \Lambda_{-v\hat{\boldsymbol{x}}}(\Lambda_{-v\hat{\boldsymbol{y}}}(0, 0)) = \Lambda_{-v\hat{\boldsymbol{x}}}(0, v) = \left(v, v\sqrt{1 - v^2}\right)$$

2.3 $t = 3\tau$

We now continue with the same reasoning. At the twice boosted frame $\Lambda_{-v\hat{\boldsymbol{x}}}\left(\Lambda_{-v\hat{\boldsymbol{y}}}\left(\cdot\right)\right)$ the velocity of the spaceship at $t = 3\tau$ is $(-v, 0)$, therefore it's velocity w.r. to the earth is:

$$\boldsymbol{v}\left(3\tau\right) = \Lambda_{-v\hat{\boldsymbol{x}}}\left(\Lambda_{-v\hat{\boldsymbol{y}}}\left(\Lambda_{v\hat{\boldsymbol{x}}}\left(0,0\right)\right)\right) = \Lambda_{-v\hat{\boldsymbol{x}}}\left(\Lambda_{-v\hat{\boldsymbol{y}}}\left(-v,0\right)\right)$$

$$= \Lambda_{-v\hat{\boldsymbol{x}}}\left(-v\sqrt{1-v^2}, v\right) = \left(\frac{v\left(1-\sqrt{1-v^2}\right)}{1-v^2\sqrt{1-v^2}}, \frac{v\sqrt{1-v^2}}{1-v^2\sqrt{1-v^2}}\right)$$

2.4 $t = 4\tau$

And again:

$$\boldsymbol{v}\left(4\tau\right) = \Lambda_{-v\hat{\boldsymbol{x}}}\left(\Lambda_{-v\hat{\boldsymbol{y}}}\left(\Lambda_{v\hat{\boldsymbol{x}}}\left(\Lambda_{v\hat{\boldsymbol{y}}}\left(0,0\right)\right)\right)\right) = \Lambda_{-v\hat{\boldsymbol{x}}}\left(\Lambda_{-v\hat{\boldsymbol{y}}}\left(\Lambda_{v\hat{\boldsymbol{x}}}\left(0,-v\right)\right)\right)$$

$$= \Lambda_{-v\hat{\boldsymbol{x}}}\left(\Lambda_{-v\hat{\boldsymbol{y}}}\left(-v, -v\sqrt{1-v^2}\right)\right) = \Lambda_{-v\hat{\boldsymbol{x}}}\left(\frac{-v\sqrt{1-v^2}}{1-v^2\sqrt{1-v^2}}, \frac{v\left(1-\sqrt{1-v^2}\right)}{1-v^2\sqrt{1-v^2}}\right)$$

if we denote

$$\alpha = \frac{1}{1-v^2\sqrt{1-v^2}}$$

$$\Lambda_{-v\hat{\boldsymbol{x}}}\left(-v\sqrt{1-v^2}\alpha, v\left(1-\sqrt{1-v^2}\right)\alpha\right) = \left(\frac{-v\sqrt{1-v^2}\alpha+v}{1-v^2\sqrt{1-v^2}\alpha}, \frac{v\left(1-\sqrt{1-v^2}\right)\alpha\sqrt{1-v^2}}{1-v^2\sqrt{1-v^2}\alpha}\right)$$

2.5 The speed at $t = 4\tau$

This part is pure algebra:

$$\alpha = \frac{1}{1-v^2\sqrt{1-v^2}}$$

$$\boldsymbol{v}\left(4\tau\right) = \left(\frac{-v\sqrt{1-v^2}\left(\frac{1}{1-v^2\sqrt{1-v^2}}\right)+v}{1-v^2\sqrt{1-v^2}\left(\frac{1}{1-v^2\sqrt{1-v^2}}\right)}, \frac{v\left(1-\sqrt{1-v^2}\right)\left(\frac{1}{1-v^2\sqrt{1-v^2}}\right)\sqrt{1-v^2}}{1-v^2\sqrt{1-v^2}\left(\frac{1}{1-v^2\sqrt{1-v^2}}\right)}\right)$$

$$= \left(\frac{-v\sqrt{1-v^2}+v\left(1-v^2\sqrt{1-v^2}\right)}{1-2v^2\sqrt{1-v^2}}, \frac{v\left(1-\sqrt{1-v^2}\right)\sqrt{1-v^2}}{1-2v^2\sqrt{1-v^2}}\right)$$

$$= \left(\frac{-v\sqrt{1-v^2}\left(1+v^2\right)+v}{1-2v^2\sqrt{1-v^2}}, \frac{v\left(\sqrt{1-v^2}-1+v^2\right)}{1-2v^2\sqrt{1-v^2}} \right)$$

$$= v\left(\frac{-\sqrt{1-v^2}\left(1+v^2\right)+1}{1-2v^2\sqrt{1-v^2}}, \frac{\sqrt{1-v^2}-1+v^2}{1-2v^2\sqrt{1-v^2}} \right)$$

$$|\boldsymbol{v}\left(4\tau\right)|^2 = v^2\frac{1}{\left(1-2v^2\sqrt{1-v^2}\right)^2}\left(\left(1-v^2\right)\left(1+v^2\right)^2+1-2\sqrt{1-v^2}\left(1+v^2\right)+\left(1-v^2\right)+1+v^4-2v^2-\right.$$

$$|\cdot|^2 = v^2\frac{1}{\left(1-2v^2\sqrt{1-v^2}\right)^2}\left(\sqrt{1-v^2}\left(-4\right)-v^6+4-2v^2\right)$$

$$|\cdot|^2 = v^2 \to \left(\sqrt{1-v^2}\left(-4\right)-v^6+4-2v^2+\right) = 1+4v^4\left(1-v^2\right)-4v^2\sqrt{1-v^2}$$

$$3-2v^2-4v^4+3v^6-4\sqrt{1-v^2}\left(1-v^2\right)=0$$

$$\boxed{\left(\left(-3v^4+3+v^2\right)-4\sqrt{1-v^2}\right)\left(1-v^2\right)=0} \qquad (1)$$

This equation has only one solution in the range $0 < v < 1$ which is $\boxed{v = 0.7862}$. Solved numerically using Desmos graphing calculator.

3 Another method

The reference frame of the spaceship at $t = 0$ is I. At $t = \tau$ it is $\Lambda\left(v\hat{x}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$. At $t = 2\tau$ the frame of reference is: $\Lambda\left(v\hat{x}\right)\Lambda\left(v\hat{y}\right) = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & 0 & -\beta\gamma \\ 0 & 1 & 0 \\ -\beta\gamma & 0 & \gamma \end{pmatrix}$. At $t = 3\tau$ it is $\Lambda\left(v\hat{x}\right)\Lambda\left(v\hat{y}\right)\Lambda\left(-v\hat{x}\right)$ and at $t = 4\tau$ it is $\Lambda\left(v\hat{x}\right)\Lambda\left(v\hat{y}\right)\Lambda\left(-v\hat{x}\right)\Lambda\left(-v\hat{y}\right)$. To find the velocity of the spaceship in the rest frame at $t = 4\tau$ we can observe how the transformation acts on a stationary point $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\Delta t$.

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & 0 & -\beta\gamma \\ 0 & 1 & 0 \\ -\beta\gamma & 0 & \gamma \end{pmatrix}\begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \gamma & 0 & \beta\gamma \\ 0 & 1 & 0 \\ \beta\gamma & 0 & \gamma \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\Delta t$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & -\beta\gamma \\ 0 & 1 & 0 \\ -\beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ 0 \\ \beta\gamma \end{pmatrix} \Delta t$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & -\beta\gamma \\ 0 & 1 & 0 \\ -\beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma^2 \\ \beta\gamma^2 \\ \beta\gamma \end{pmatrix} \Delta t$$

$$= \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma^3 - \beta^2\gamma^2 \\ \beta\gamma^2 \\ -\beta\gamma^3 + \beta\gamma^2 \end{pmatrix} \Delta t$$

$$= \begin{pmatrix} \gamma^4 - \beta^2\gamma^3 - \beta^2\gamma^3 \\ -\beta\gamma^4 + \beta^3\gamma^3 + \beta\gamma^3 \\ -\beta\gamma^3 + \beta\gamma^2 \end{pmatrix} \Delta t$$

Therefore the velocity in the rest frame is:

$$v_x = \frac{-\beta\gamma^4 + \beta^3\gamma^3 + \beta\gamma^3}{\gamma^4 - 2\beta^2\gamma^3} = \beta\frac{\beta^2 + 1 - \gamma}{\gamma - 2\beta^2}$$

$$v_y = \frac{\beta}{\gamma}\frac{1-\gamma}{\gamma - 2\beta^2}$$

And the condition is $v_x^2 + v_y^2 = v^2 = \beta^2$ so:

$$\frac{\beta^2}{\gamma^2}\left(1 - 2\gamma + \gamma^2\right) + \beta^2\left(\beta^4 + 1 + \gamma^2 - 2\gamma - 2\gamma\beta^2 + 2\beta^2\right) = \beta^2\left(\gamma^2 - 4\gamma\beta^2 + 4\beta^4\right)$$

$$\frac{1}{\gamma^2}\left(1 - 2\gamma + \gamma^2\right) - 3\beta^4 + 1 - 2\frac{1}{\gamma} + 2\beta^2 = 0$$

$$-3\beta^4 + \beta^2 + 3 - 4\frac{1}{\gamma} = 0$$

Which is the same as equation 1.