Physics Cup Problem 4

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1 Interpretation of the Problem

The problem states that the magnitude of the proper acceleration is constant ($= g$), but the direction changes by 90 degrees at specified proper-times, and this happens via rotating the engine (or some steering wheel within the spaceship). So that means that the 90 degree turn happens in the instantaneous rest frame of the spaceship, not in the earth frame. We also assume that this happens abruptly.

2 Getting started

First let us change the notation a bit:

- Use λ for τ
- Use τ for proper time of the spaceship

Thus, the spaceship has its proper acceleration constant (in direction too) for $\tau \in [n\lambda, (n+1)\lambda)$ for $n = (0, 1, 2, 3)$, and changes abruptly at $\tau = n\lambda$.

Let us look at the spaceship from the Earth's frame S during the time $\tau \in [0, \lambda)$. What is the speed of the spaceship u, and its gamma factor γ, at $\tau = \lambda$ in frame S? This is a fairly easy (textbook type) problem, and the solution (using natural units) is [2]

$$u = \tanh(g\lambda) \tag{1}$$

$$\gamma = \cosh(g\lambda) \tag{2}$$

3 General Idea for the full solution

We can map parts of the full problem to many instances of the above simple problem and piece the solutions together to get our answer. The way to do this is to have 4 inertial frames $S, S^{'}, S^{''}, S^{'''}$ which are the frames in which the spaceship is instantaneously at rest at proper times $0, \lambda, 2\lambda, 3\lambda$ respectively. In each of these frames we look at the spaceship for proper time λ from the time it is at rest. The velocities and gamma factors are all the same as what is calculated above! Then we just need to do some velocity transformations to find the velocity of the spaceship at proper time 4λ in frame S. This can be done in at least two ways:

- Since the velocity of the spaceship (at proper time 4λ) in frame $S^{'''}$ is trivially known - call it $\vec{u}^{'''}$ - and since the velocity of the frame S''' in frame $S^{''}$ is trivially known - call it $\vec{v}^{''}$ - we can use velocity addition to find the velocity of the spaceship in frame $S^{''}$, $\vec{u}^{''}$. Then we can forget about frame $S^{'''}$ and iteratively proceed ("backwards") and get the velocity of the spaceship in frame S, $\vec{u}$.
- Do it the opposite way: given the relation between S and $S^{'}$ and the relation between $S^{'}$ and $S^{''}$, relate S and $S^{''}$. Then forget $S^{'}$ and proceed ("forwards") iteratively and get the velocity of the spaceship in frame S, $\vec{u}$.

While both methods seem similar at first sight, the first way is significantly easier to do because we don't have to worry about Thomas-Wigner rotations!

4 What's with the Wigner Rotations?

One can find a detailed account of Wigner Rotations in e.g. Steane's book [1]. Essentially, in a setup like above, the frames $(S, S^{'})$ have their spatial coordinate axes aligned (in frames S and $S^{'}$) and the same is true for the pair $(S^{'}, S^{''})$. However the frames $(S, S^{''})$ are *not* aligned, in both frames S and $S^{''}$. Instead they are actually rotated with respect to each other. The reason this happens is that in general the composition of two boosts is equal to a boost and a rotation. (The rotation goes away only when the two boosts are aligned, which is not the case in our problem!)

Thus, if we follow method 2, we will have to first undo the rotation of $S^{''}$ with respect to S, making the frames aligned, before applying the standard velocity addition formula. This has to be done at every step of the forward iteration. However, if we follow method 1, this problem never arises, since we never have to apply velocity addition between two non-consecutive frames!

Note that if we consider the spaceship to also be a reference frame then we have not undone the rotation of that with respect to S at the end, but we don't really care about the orientation of the spaceship - we only need its velocity in frame S.

5 The Solution

Here we implement the first (easy) solution. We adopt the natural notation of using $\vec{u}^n$ for the velocity of the spaceship (at $\tau = 4\lambda$) in frame S^n and $\vec{v}^{n-1}$ for the velocity of frame S^n in frame S^{n-1}. (Here the power n is the number of primes)

We have

$$\vec{u}^{'''} = -\tanh(g\lambda)\hat{y} \tag{3}$$

$$\vec{v}^{''} = -\tanh(g\lambda)\hat{x} \tag{4}$$

Note how we use the same unit vectors. This is because the equation is written in frame $S^{''}$ whose axes are parallel to frame $S^{'''}$! When we proceed "backwards" like this, the coordinate axes being aligned will be taken care of automatically, so from now we won't bother to be super careful. From this we get

$$\vec{u}^{''} = -\tanh(g\lambda)(\hat{x} + \frac{\hat{y}}{\cosh(g\lambda)}) \tag{5}$$

using velocity addition. Now using $\vec{v}^{'} = \tanh(g\lambda)\hat{y}$ we get

$$\vec{u}^{'} = \frac{\tanh(g\lambda)}{\cosh(g\lambda) - \tanh(g\lambda)^2}(-\hat{x} + (\cosh(g\lambda) - 1)\hat{y}) \tag{6}$$

Repeating the velocity addition using $\vec{v} = \tanh(g\lambda)\hat{x}$ and simplifying some algebra we get

$$\vec{u} = \frac{\tanh(g\lambda)}{\cosh(g\lambda) - 2\tanh(g\lambda)^2}((\cosh(g\lambda) - \tanh(g\lambda) - 1)\hat{x} + (\cosh(g\lambda) - 1)\hat{y}) \tag{7}$$

We need the magnitude of this to be equal to the magnitude of $\vec{v}$. This gives

$$g\lambda = \ln \frac{1 + \sqrt{5} + \sqrt{2(1 + \sqrt{5})}}{2} \tag{8}$$

From this we get the speed of the spaceship w.r.t Earth at proper times $\lambda, 4\lambda$

$$v_{rel} = \tanh(g\lambda) = \sqrt{\frac{\sqrt{5} - 1}{2}}c \approx 0.7862c \tag{9}$$

References

[1] Andrew M. Steane. *Relativity made Relatively Easy Vol. 1.* 2012. URL: `https://global.oup.com/academic/product/relativity-made-relatively-easy-9780199662869?lang=en&cc=inm` (visited on 03/07/2023).

[2] Jeff Tseng. *B2: Symmetry and Relativity.* 2018. URL: `http://www-pnp.physics.ox.ac.uk/~tseng/teaching/b2/b2-lectures-2018.pdf` (visited on 03/07/2023).