Problem 4 - Spaceship

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Let's use $x - ict$ coordinates. According to proofs of Physics Cup 2018 P2, the world line of the spaceship during one period is now a circular section (all of the same radius). Hence one such period of τ will give a certain rotation angle to the ship in x-ict coordinates. Since there is movement in two different directions, there will be rotations in both directions, on the momentary axes. For example, after the first rotation, that happened in the plane x-ict, the ship is at point P. The next rotation will now happen in the plane, containing the y axis and point P, since the momentary $ic\tau$ axis goes through P and is perpendicular to y axis.

Now we will use the intrinsic active rotation formulae to get an equation for rotation angle θ. Rotation around the x axis on a point (x,y,z) with angle θ is defined as

$$(x, y, z) \mapsto (x, y\cos(\theta) - z\sin(\theta), y\sin(\theta) + z\cos(\theta))$$

So now we take an arbitrary (sort of) point (0,0,1), that represents the starting direction of movement and apply the 4 transformations:

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\-\sin\varphi\\\cos\varphi\end{bmatrix} \mapsto \begin{bmatrix}\sin\varphi\cos\varphi\\-\sin\varphi\\\cos^2\varphi\end{bmatrix} \mapsto \begin{bmatrix}\sin\varphi\cos\varphi\\-\sin\varphi\cos\varphi+\sin\varphi\cos^2\varphi\\\sin^2\varphi+\cos^3\varphi\end{bmatrix} \mapsto$$

$$\mapsto \begin{bmatrix}\sin\varphi\cos^2\varphi - \sin^3\varphi - \sin\varphi\cos^3\varphi\\-\sin\varphi\cos\varphi+\sin\varphi\cos^2\varphi\\\sin^2\varphi\cos\varphi+\sin^2\varphi\cos\varphi+\cos^4\varphi\end{bmatrix} \tag{1}$$

Since only rotations were applied, all the vectors should have the same length. Hence the distance between the initial vector and the first one should be the same as the distance from the initial vector to (1). Writing down the equality of these differences(with $s = \sin\varphi$ and $c = \cos\varphi$ for convenience while bashing):

Distance after first rotation:

$$(-s)^2 + (c-1)^2 = s^2 + c^2 - 2c + 1 = 2(1-c)$$

After all the rotations:

$(sc^2 - s^3 - sc^3)^2 + (-sc + sc^2) + (2s^2c + c^4 - 1)^2 = s^2(c^2 - s^2 - c^3)^2 + s^2c^2(-1+c)^2 + (c(2s^2+c^3)-1)^2 =$
$s^2c^4 + s^6 + s^2c^6 + 2s^4c^3 - 2s^4c^2 - 2s^2c^5 + s^2c^2 + s^2c^4 - 2s^2c^3 + 4s^4c^2 + c^8 + 4s^2c^5 - 4s^2c - 2c^4 + 1 =$
$c^8 + s^2c^6 + 2s^4c^3 + 2s^2c^5 + s^6 + 2s^4c^2 + 2s^2c^4 - 2s^2c^3 + s^2c^2 - 2c^4 - 4s^2c + 1 =$
$c^6 + 2s^2c^3 + s^6 + 2s^2c^2 - 2s^2c^3 + s^2c^2 - 2c^4 - 4s^2c + 1 =$
$c^4 - s^2c^2 + s^4 + 3s^2c^2 - 2c^4 - 4s^2c + 1 =$

$1 - 2c^4 - 4s^2c + 1 =$
$2 - 2c^4 - 4c + 4c^3$

Equating these two distances (squared):
$2 - 2c^4 - 4c + 4c^3 = 2(1-c) \Rightarrow$

$$c^4 - 2c^3 + c = 0 \Rightarrow c(c-1)(c^2 - c - 1) = 0$$

This offers us trivial zeros $\cos\varphi = 0, \cos\varphi = 1$ and some more interesting ones:

$$\cos\varphi = \frac{1 \pm \sqrt{5}}{2}$$

Since rotations in the $x - ict$ space-time require an imaginary angle, the only valid solution is the golden ratio: $\frac{1+\sqrt{5}}{2}$. Using some trig formulae: $\cos i\theta = \cosh\theta \Rightarrow \phi = i\cosh^{-1}\frac{1+\sqrt{5}}{2}$

Now in the $x - ict$ plane, an imaginary rotation angle φ satisfies the following condition: $\tan\varphi = \frac{v}{ic} \Rightarrow \tanh\frac{\varphi}{i} = \frac{v}{c}$. Leading to our final equation:

$$v = c\tanh\cosh^{-1}\frac{1+\sqrt{5}}{2} \approx 0.786151378c$$

Looking forward to solving this in a normal (not as bashy) way. Also working on sketching the 3d space-time for this trip of the spaceship.