Physics Cup 2023 Problem 5

Eppu Leinonen

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Contents

1 Introduction

First, I will give a brief solution explaining the physics and the description of the sources. Thereafter, I will prove a certain geometrical construction based on the given tools in GeoGebra, and finally give the steps in GeoGebra to construct the source points.

2 Solution

Let's look at two of the light sources. As we have coherent light sources, the light from two of the sources will interfere constructively at P when the path difference is an integer multiple. If the distance of the first light source, A, to P is d_{AP} and from the second light source, B, to P is d_{BP}, the condition can be stated mathematically as:

$$|d_{AP} - d_{BP}| = n\lambda, n \in \mathbb{N}_0.$$

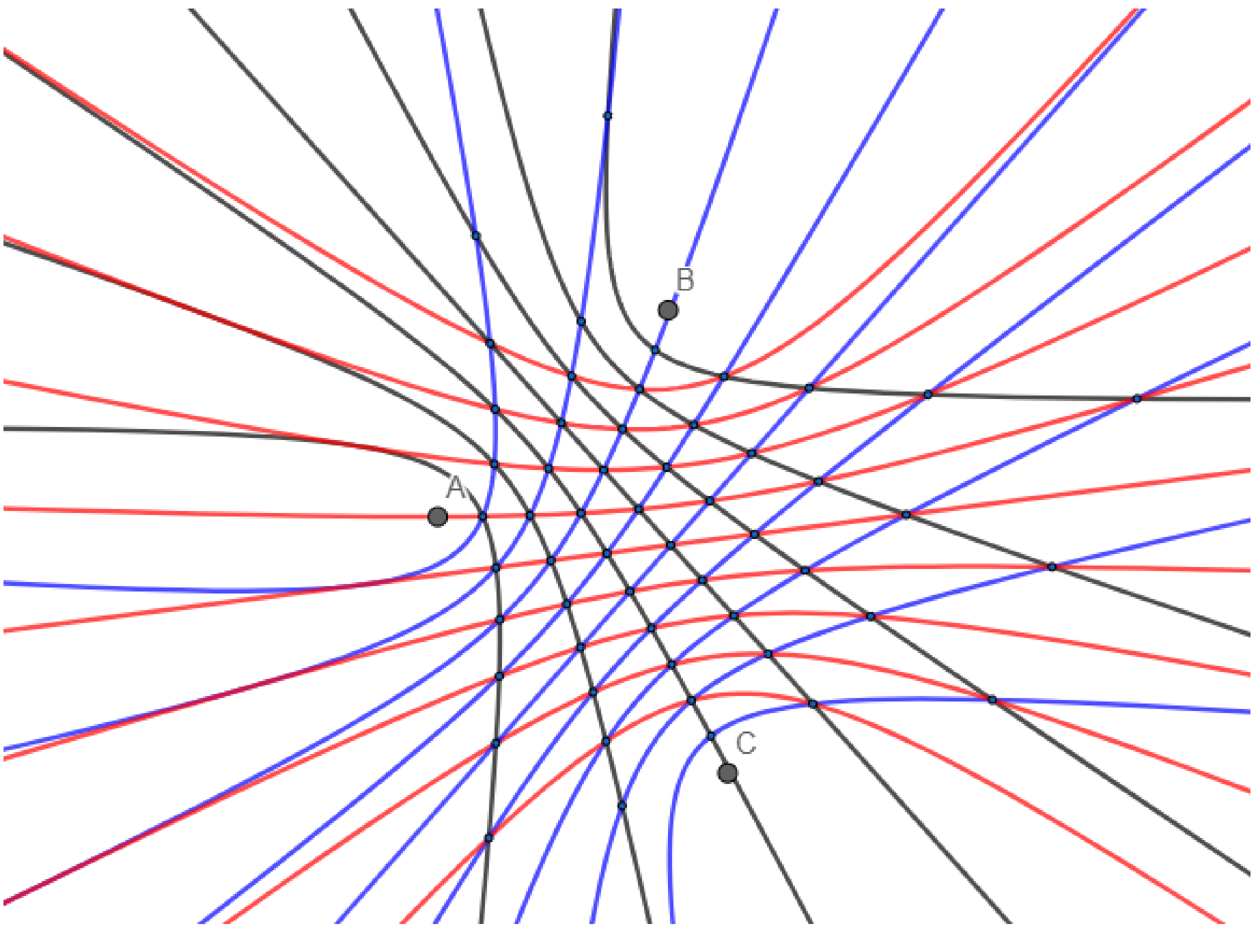
For fixed n, these points form a hyperbola with foci A and B. (The case $n = 0$ referring to a line, which we will interpret as a limit of a hyperbola). As a mathematical side note, there can only be a limited number of these hyperbola as the maximum value of the path difference is the physical distance between the sources.

Now, when we introduce the third light source, C. It will form more of these hyperbolas with both of the points. Hence, there will be hyperbola with foci $A/B, A/C, B/C$. At any intersection of these hyperbolas, all of the light from the light sources will be at phase. Respectively, any deviation from this intersection will lead to a point where they will be slightly out of phase and hence lead to a loss of maximal amplitude and hence a loss of intensity as $I \propto A^2$. Thus, these intersection points are local maxima of intensity.

Now given the maxima, we can just find the respective hyperbola and construct their foci to be the light sources. (It is to be noted, that there is a high chance that any 5 particular points form a conic section of a hyperbola which is not related to the phase defined hyperbola above. Therefore technically, one find the straight lines going through 5 points which must be the conjugate axes of the hyperbolae and then draw the hyperbolae). This will yield (rounded to 3 decimal point):

$$A = (-6.000, -2.000) \qquad B = (-0.041, 3.338) \qquad C = (1.497, -8.618)$$

Here is a picture of the hyperbolae and the foci. The hyperbolae defined by A and B are black, by A and C blue, and by B and C red.


A
B
C

3 Construction of the foci of a hyperbola using GeoGebra

There is the command `focus()` which outputs the foci of any conic section. However, I will show a way to construct the foci of a hyperbola given the tools in the graphics window of GeoGebra.

First, we will derive the following theorem:

Theorem. *Let the foci of a hyperbola be F_1 and F_2 and O the centre of symmetry of the hyperbola. Let T be the intersection between the transverse axis and the tangent drawn at some point P of the hyperbola and N be the respective intersection of the transverse axis and the normal at P. Then $OF_1^2 = OT \cdot ON$.*

Proof. Without loss of generality, let us orient the hyperbola such that its transverse axis lies on the x-axis of a Cartesian coordinate system. Then the general form the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Now OF is known to be $a^2 + b^2$.

The well-known formulae for the tangent and normal drawn at the point (x_1, y_1) on the hyperbola are

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1 \qquad \text{and} \qquad a^2 y_1 x + b^2 x_1 y - (a^2 + b^2) x_1 y_1 = 0.$$

Thus, OT and ON are the values of x at $y = 0$. I.e.

$$\frac{x_1 \cdot OT}{a^2} = 1$$
$$\implies \qquad OT = \frac{a^2}{x_1}$$

and

$$a^2 y_1 \cdot ON - (a^2 + b^2) x_1 y_1 = 0$$
$$\implies a^2 \cdot ON - OF^2 \cdot x_1 = 0$$
$$\implies ON = \frac{OF^2 \cdot x_1}{a^2}.$$

Hence,

$$OT \cdot ON = \frac{a^2}{x_1} \cdot \frac{OF^2 \cdot x_1}{a^2} = OF^2.$$

□

In other words the distance of the foci from the centre of symmetry is the geometric mean of the tangent and normal intersection lenghts. Thus for our construction to work we have to be able to form the geometric mean geometrically.

To do this, let O, A, and B be points in order on a line. Let P be a point such that $PO = PA$ and $OB = PB$. Thus the triangles $\triangle OPA$ and $\triangle OBP$ are isosceles with base angle $\angle BOP$ and hence similar. From the similarity we get,

$$\frac{PO}{OP} = \frac{OA}{PO} \implies PO^2 = OA \cdot OB.$$

I.e. PO is the geometric mean of OA and OB. We can use this to construct a circle of radius $\sqrt{OA \cdot OB}$ given the point O, A, B.

Let O, A, and B be points in order on a line. If we draw a circle of radius OB with the centre B and a circle of the same radius around A, the circle of radius OB drawn around the intersection point of the A-centric circle and the line OAB, I, will intersect the B-centric circle at P. Now $BP = BO$ as points P and O are on the same B-centric circle, and the symmetry of the construction means that $OP = AP$. Hence the situation is the same as before and $OP = \sqrt{OA \cdot OB}$ and the circle of radius OP is what we were after.

Back to the hyperbola — using GeoGebra we can can construct the centre of symmetry O using the center tool on the hyperbola. Next we can pick a point P on the hyperbola. Using the tangent tool, we can draw the tangent of the hyperbola at P and similarly using the perpendicular tool we can construct the normal at P. The transverse axis is able to be constructed using GeoGebras extremum tool on the hyperbola to give its vertices and connecting them with the line tool. Now, T is simply the intersection between the tangent and the line V_1V_2 and N between the normal and V_1V_2.

We can use the above construction on points O, T, N (respective for O, A, B in the construction) using the circle and compass tools. This will give us the point R and drawing the circle of radius OR at O will give us the foci as the intersections of the circle and the line V_1V_2 based on the proven theorem.

Also, if one focus is known, the other can be constructed by mirroring the first focus about the centre of symmetry. (This will be used to construct the third point.)

4 All the steps in GeoGebra to the solution

Using the Conic through 5 Points tool, draw a hyperbola, c, through the points $G1, M1, P1, L1, J1$.

Using the Extremum and Midpoint or Center tools, construct the vertices V_1 and V_2 and the centre O of c.

Draw the line, f, through V_1 and V_2.

Pick an arbitrary point P on c and draw the respective tangent, g, using the Tangents tool.

Draw a normal, h, to g at P using the Perpendicular line tool.

Using the Intersect tool, draw the intersection, T, between g and f and the intersection, N, between h and f.

Draw the circle, d, with the centre N and point O with the Circle with Center through Point tool. Similarly, using the Compass tool, draw a circle, e, with the centre P and radius NO.

Draw the intersection, I, between e and f using the Intersect tool.

Draw the circle, K, with the centre I and radius NO using the Compass tool.

Using the Intersect tool, draw one of the intersections, R, between k and e. Draw the circle, p, with the centre O and point R using the Circle with Center through Point tool.

Using the Intersect tool, draw the intersections, A and B, between p and f. These are two of the light sources.

Using the Conic through 5 Points tool, draw the hyperbola, q, through $P1, C2, Q2, D3, X2$.

Using the Extremum tool, draw the vertices, V_3 and V_4, of q.

Draw the transverse axis, i, of q by using the Line tool on V_3 and V_4. This goes through B so B must be one of the foci of q.

Draw the centre of q, S, by using the Midpoint or Center tool on q.

Reflect point B about S using the Reflect about Point tool. This is the third light source.