Problem 5 – Light sources

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Contents

1 Task

There are three identical point light sources emitting mutually coherent light homogeneously in all directions. The figure below shows the positions of the local light intensity maxima in the plane defined by the three light sources. These positions are not perfectly accurate because they are calculated on the assumption that the light intensity from a single source does not depend on the distance from it. Reconstruct the positions of the light sources using GeoGebra classic; you can use all its construction tools except the ones which require inputting numerical values or formulas.

2 Interference of two circular waves

First we consider the interference of the light waves emanating from two such identical light sources at points A and B. The phases with which the light waves are emitted at a certain time differ by α. Their direction of polarization is the same. Assuming that the intensity of a wave does not depend on the distance from the source, the electric field at a certain point P at a certain time is given by

$$E = A \cdot e^{i(k\cdot|PA|-\omega t)} + A \cdot e^{i(k\cdot|PB|-\omega t+\alpha)}.$$

Here $|PA|$ and $|PB|$ are the distances of the point to the two light sources. The reason that A, k and ω are the same is that the two light sources are identical. The phase difference between the two waves at point P is $\Delta\varphi = k\left(|PB| - |PA|\right) + \alpha$. Therefore,

$$E = A \cdot e^{i(k\cdot|PA|-\omega t)} \cdot \left(1 + e^{i\Delta\varphi}\right).$$

The intensity $I \propto |E|^2 = E\overline{E}$, where $\overline{E}$ is the complex conjugate of E. It follows that

$$I \propto 1 + \cos\Delta\varphi.$$

I becomes maximal if $\Delta\varphi = 2\pi n$, where $n \in \mathbb{Z}$. The set of points

$$H_n = \left\{P \mid |PB| - |PA| = \frac{2\pi n - \alpha}{k}\right\},$$

where this is satisfied, creates, for a fixed n, a branch of a hyperbola whose foci are A and B. Its semi major axis is $\left|\frac{2\pi n-\alpha}{2k}\right|$ and its linear eccentricity equals $\frac{|AB|}{2}$. n cannot take on any value, but

$$\left|\frac{2\pi n - \alpha}{k}\right| \leq |AB|$$

must apply. This follows directly from the triangle inequality $|PB| \leq |PA| + |AB|$ and $|PA| \leq |PB|+|AB|$. If there is an $n \in \mathbb{Z}$ such that $|AB| = \frac{2\pi n-\alpha}{k}$ (1) or $|AB| = -\frac{2\pi n-\alpha}{k}$ (2), then constructive interference also occurs on the the ray starting at A and heading away from B (1) or on the the ray starting at B and heading away from A (2). If $\alpha = 0$, the intensity becomes maximal for $n = 0$ on the perpendicular bisector g of the line segment AB. In this case H_{-n} is the mirror image of H_n in g. The distance of the vertex of H_n to g is $\frac{\pi|n|}{k}$ (semi major axis).

3 Interference of three circular waves

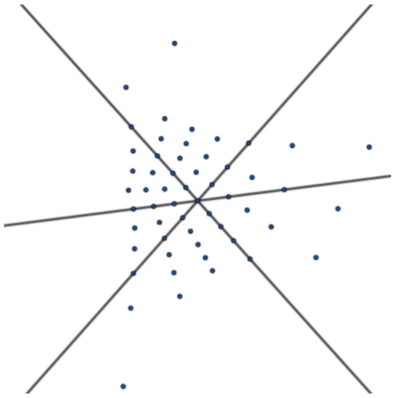
Now a third light source is added at point C. We consider the three sets of hyperbolic branches on which the intensity would become maximum if there were only two light sources.

$$M_1 = \left\{\left\{P \mid |PB| - |PA| = \frac{2\pi n_1 - \alpha_1}{k}\right\} \mid n_1 \in \mathbb{Z} \text{ and } \left|\frac{2\pi n_1 - \alpha_1}{k}\right| \leq |AB|\right\}$$
$$M_2 = \left\{\left\{P \mid |PC| - |PB| = \frac{2\pi n_2 - \alpha_2}{k}\right\} \mid n_2 \in \mathbb{Z} \text{ and } \left|\frac{2\pi n_2 - \alpha_2}{k}\right| \leq |BC|\right\} \text{ and}$$
$$M_3 = \left\{\left\{P \mid |PC| - |PA| = \frac{2\pi n_3 - (\alpha_1+\alpha_2)}{k}\right\} \mid n_3 \in \mathbb{Z} \text{ and } \left|\frac{2\pi n_3 - (\alpha_1+\alpha_2)}{k}\right| \leq |AC|\right\}$$

α_1, α_2 and $\alpha_1 + \alpha_2$ are the phase differences between the light sources at A and B, B and C and A and C respectively. At any intersection point $Q \in H_{n_1} \in M_1$ and $Q \in H_{n_2} \in M_2$, the phase difference between the two waves emanating from A and B is a multiple of 2π: $\varphi_B - \varphi_A = 2\pi n_1$. Similarly: $\varphi_C - \varphi_B = 2\pi n_2$. Therefore, $\varphi_C - \varphi_A = 2\pi\left(n_1 + n_2\right)$. Thus there is also a curve from M_3 that contains Q. Here, $n_3 = n_1 + n_2$. It follows: There is no point at which only two hyperbolas/ straight lines/ rays from M_1, M_2 and M_3 cross. At all such points of intersection, the amplitude of the electric field $|E| = 3A$ and thus also the intensity I takes on the maximum possible value. Furthermore, there are no other positions where the intensity has a local maximum (see 5.3 for proof).

4 Determining the positions of the light sources

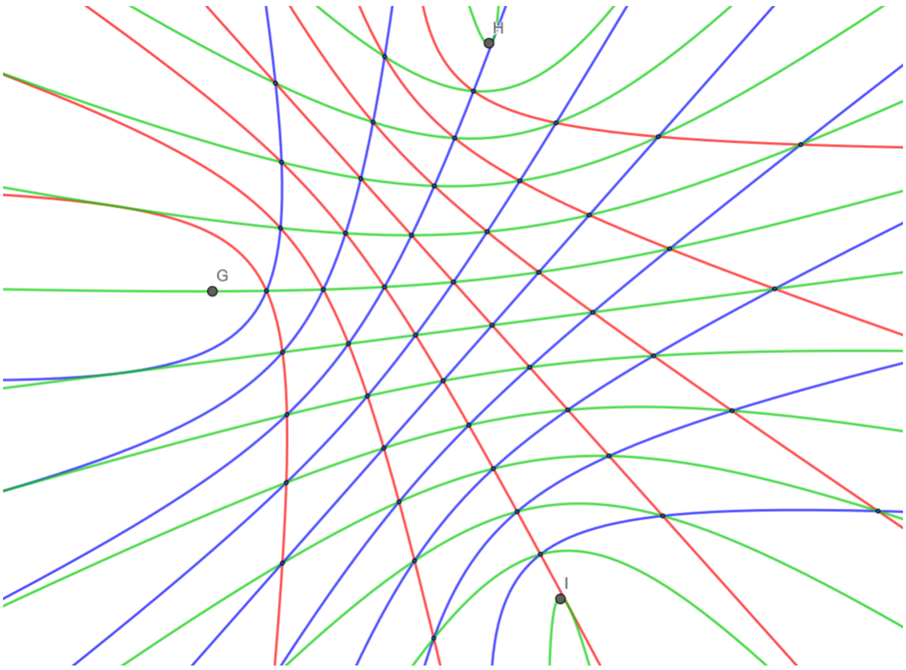
Looking more closely at the intensity maxima, we find three straight lines running through several points.



Furthermore, we can find hyperbolas that pass through some intensity maxima. For this we use the GeoGebra function “Conic through 5 Points”. The resulting sets of curves M_1, M_2 and M_3 are clearly recognisable (see 5.1). The straight lines are the perpendicular bisectors of the line segments between the points at which the light sources are located. These therefore emit the light waves in phase ($\alpha_1 = \alpha_2 = 0$).

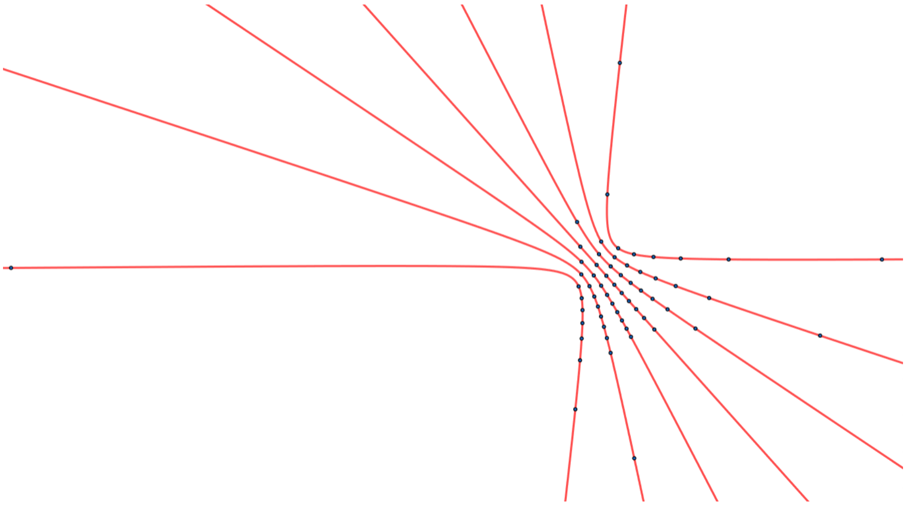
We know that the light sources are located at the focal points of the hyperbolas. To find them, we draw for $i = 1, 2, 3$ each the axis of symmetry of the hyperbolic branches $H_n \in M_i$ ($n \neq 0$), which is perpendicular to the corresponding straight line $H_0 \in M_i$. The algorithm to do this can be found in the appendix (5.2). The resulting points of intersection represent the focal points. They are $(-6, -2)$ (G), $(-0.041520, 3.338213)$ (H) and $(1.497167, -8.617589)$ (I).

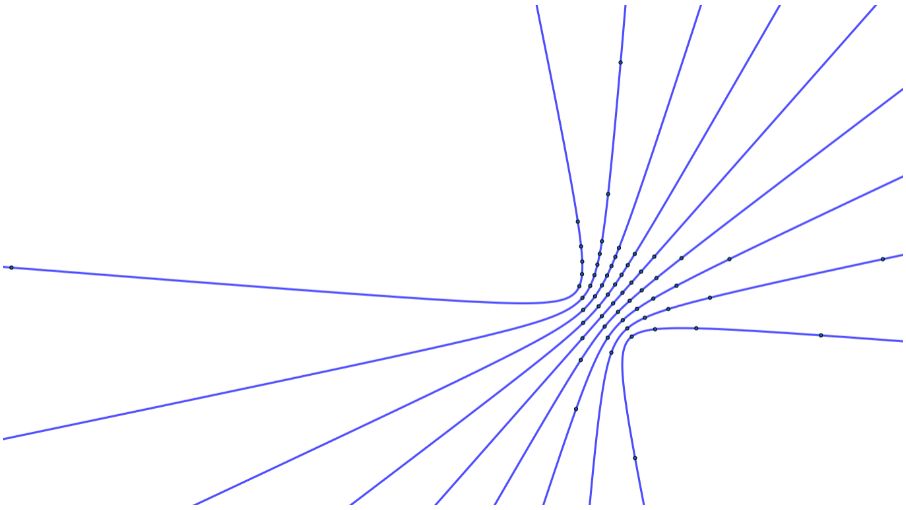
Now we can create another hyperbola that has the focal points H and I and passes through the three remaining free points (see 5.1, bottom figure). In fact, three curves from M_1, M_2 and M_3 intersect in each intensity maximum.

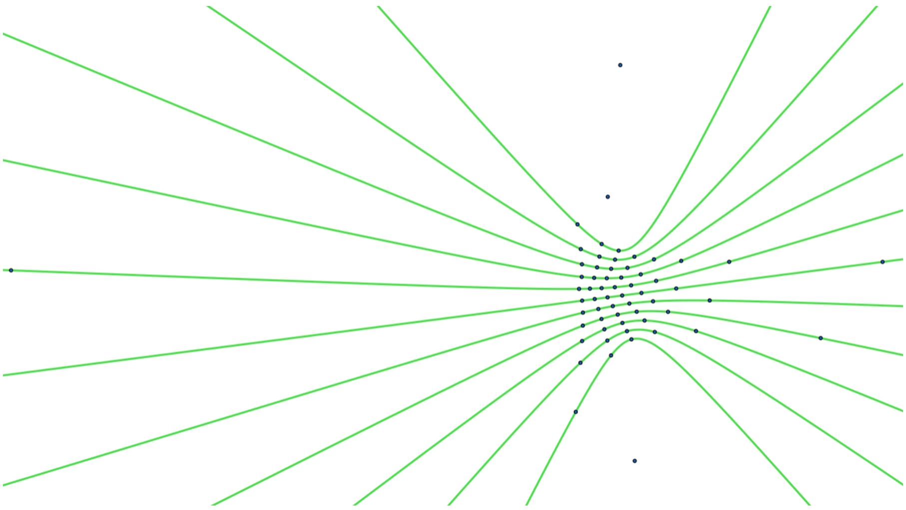


5 Appendix

5.1 Three sets of hyperbolic curves

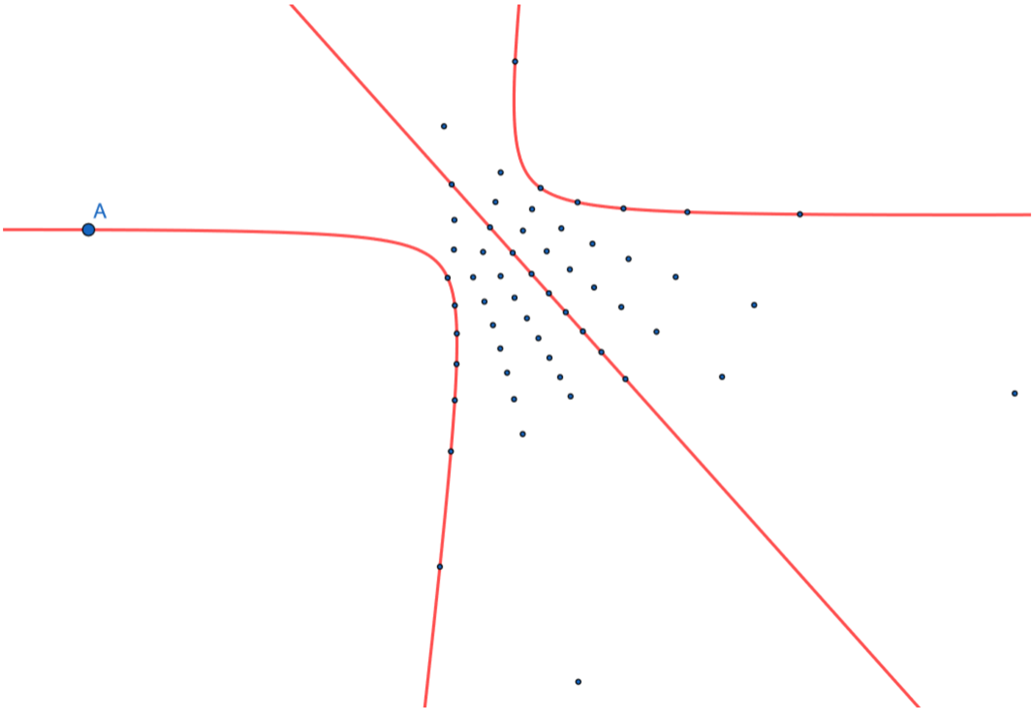




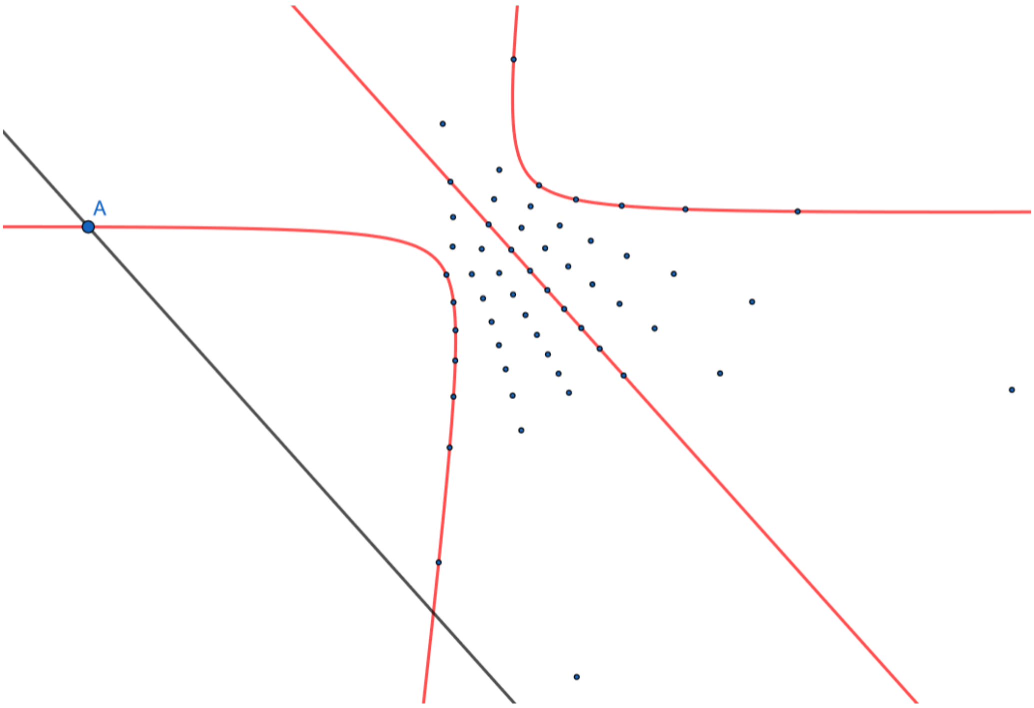


5.2 Constructions in GeoGebra to find the symmetry axis

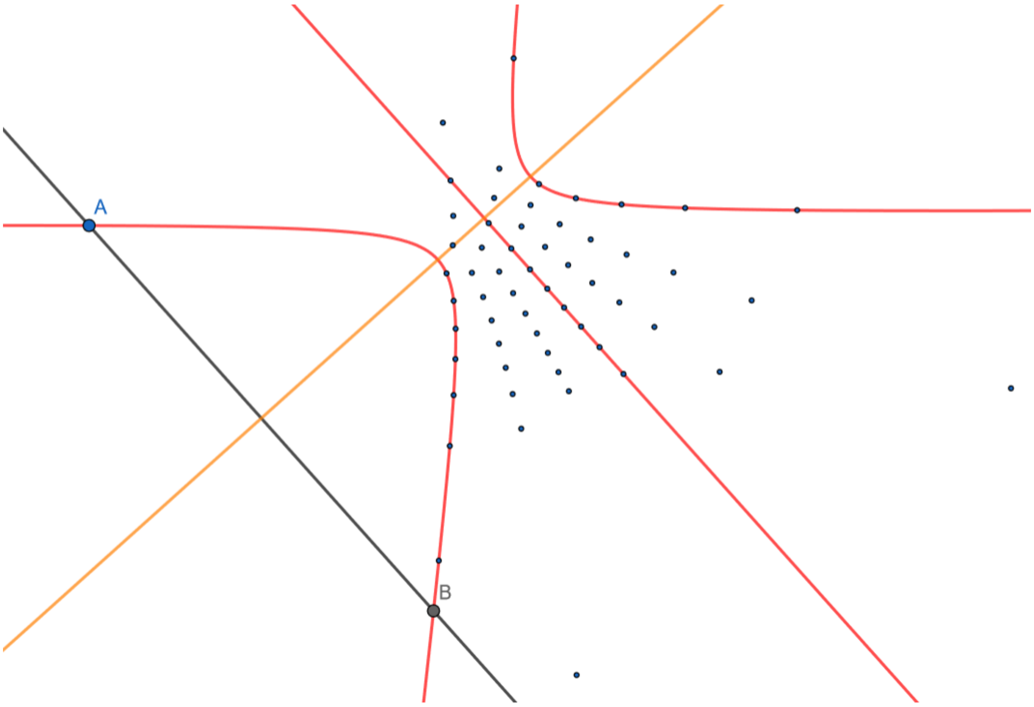
1. Choose a random point A on a hyperbolic branch from M_1:



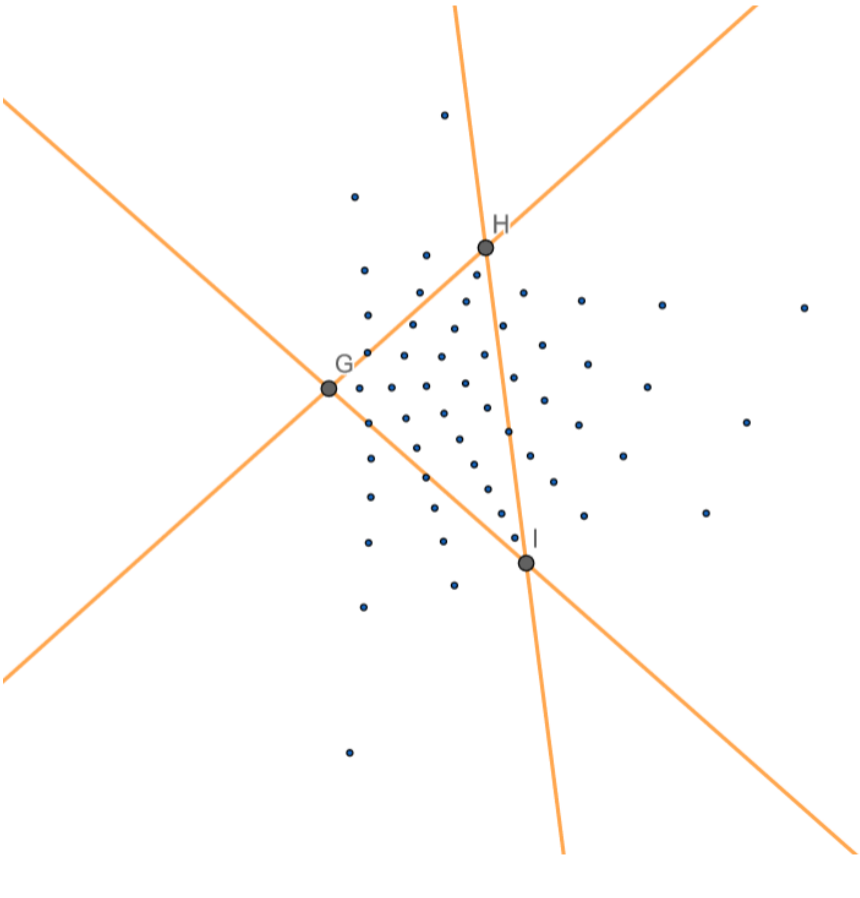
2. Draw a straight line through this point which is parallel to $H_0 \in M_1$.



3. This straight line intersects the hyperbolic branch at point B. Now plot the perpendicular bisector of the line segment AB. This is the symmetry axis we are looking for.



4. Repeat this for $i = 2$ and $i = 3$. We contain three straight lines that intersect at the sought positions of the light sources:



5.3 The reason why there are no further local maxima

In the following it will be shown that the intensity only has local maxima at the points where it takes on the maximum possible value.

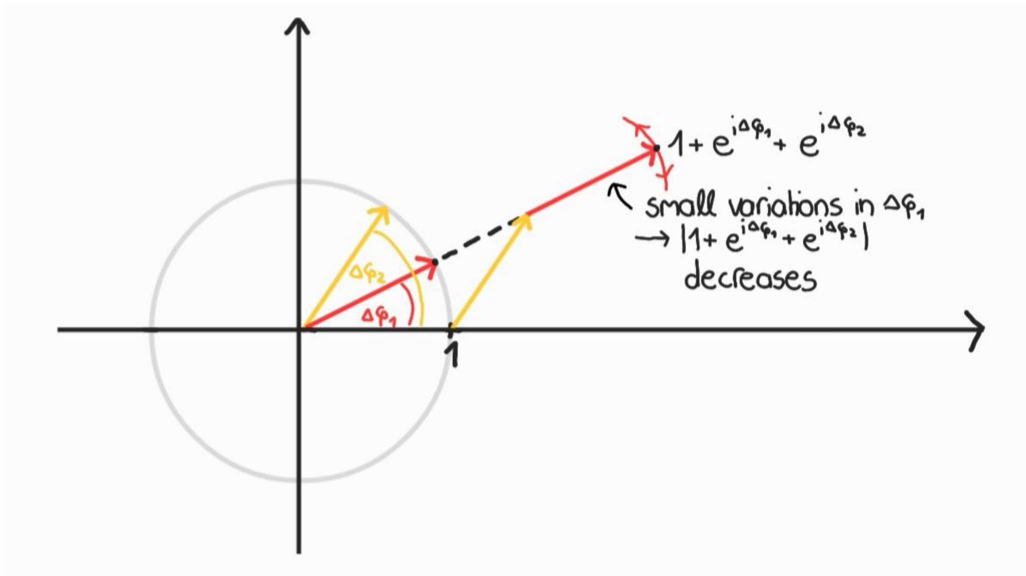
The magnitude of the electric field at a point P

$$|E| \propto |1 + e^{i\Delta\varphi_1} + e^{i\Delta\varphi_2}|,$$

where $\Delta\varphi_1 = \varphi_B - \varphi_A$ and $\Delta\varphi_2 = \varphi_C - \varphi_A$ are the phase differences between the two waves emanating from A and B and A and C respectively. We consider the two hyperbolic branches passing through P on which $\Delta\varphi_1 = \text{const.}$ or $\Delta\varphi_2 = \text{const.}$ holds. Assuming P is a local maximum, $|E|$ should decrease for small displacements along each of these hyperbolas, which corresponds to small changes in $\Delta\varphi_1$, where $\Delta\varphi_2 = \text{const.}$ and vice versa. This is the case if

$$1 + e^{i\Delta\varphi_1} + e^{i\Delta\varphi_2} = k_1 \cdot e^{i\Delta\varphi_1},$$

where $k_1 > 1$, as can be understood with the help of the following figure.



Similarly,

$$1 + e^{i\Delta\varphi_1} + e^{i\Delta\varphi_2} = k_2 \cdot e^{i\Delta\varphi_2},$$

where $k_2 > 1$. From $k_1 \cdot e^{i\Delta\varphi_1} = k_2 \cdot e^{i\Delta\varphi_2}$ it follows that $\Delta\varphi_1 = \Delta\varphi_2 + 2\pi n$, where $n \in \mathbb{Z}$ (and $k_1 = k_2$). Then from $1 = (k_1 - 2) \cdot e^{i\Delta\varphi_1}$ we obtain $\Delta\varphi_1 = 2\pi m$, where $m \in \mathbb{Z}$ (as well as $k_1 = k_2 = 3$). Therefore, if there is a local maximum of intensity at point P, then the three waves emanating from the light sources interfere having the same phase, so that I takes on the largest possible value.