Physics Cup P5

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§1 Initial analysis

This is simply the interference due to three point light sources, say with wavelength δ. Let's say at a point P, and if the phases of the three light beams are ϕ, Φ, φ, then the theoretical maximum of intensity (global and local) is achieved when $\phi - \Phi = 2\pi k_1$ and $\Phi - \varphi = 2\pi k_2$, where k_1 and k_2 are integers. Obviously, for fixed k_1 and k_2, there are 2 degrees of constraints here. However, in the plane, there are 2 degrees of freedom, so there could be a solution corresponding to the equations $\phi - \Phi = 2\pi k_1$ and $\Phi - \varphi = 2\pi k_2$ for some values of k_1 and k_2. In other words, this theoretical maximum of all the phases being equal could be achievable.

§1.1 But can there be local maxima that are not global?

This is an important question to consider. We will consider the local maxima of $\vec{r}$ with respect to light sources $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$. But this means we would like to maximize the amplitude of

$$e^{i|\vec{r}-\vec{r_1}|} + e^{i|\vec{r}-\vec{r_2}|} + e^{i|\vec{r}-\vec{r_3}|}$$

or in other words that a small perturbment of $\vec{r}$ will leave the amplitude of this unchanged. To do this, we would need to consider $|\vec{r} + \vec{\delta r} - \vec{r'}| - |\vec{r} - \vec{r'}|$. In the First Appendix, we prove that this is $\vec{\delta r} \cdot \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|}$ We call the unit vector $\frac{\vec{r}-\vec{r_1}}{|\vec{r}-\vec{r_1}|}$ as $\vec{e_1}$. Thus, if $\vec{r}$ becomes $\vec{r} + \vec{\delta r}$, then the

$$e^{i|\vec{r}-\vec{r_1}|} + e^{i|\vec{r}-\vec{r_2}|} + e^{i|\vec{r}-\vec{r_3}|}$$

changes by

$$i\vec{r_1} \cdot \vec{\delta r} e^{i|\vec{r}-\vec{r_1}|} + \vec{r_2} \cdot \vec{\delta r} e^{i|\vec{r}-\vec{r_2}|} + \vec{r_3} \cdot \vec{\delta r} e^{i|\vec{r}-\vec{r_3}|}$$

which is then rewritten as

$$i\left(\vec{r_1} e^{i|\vec{r}-\vec{r_1}|} + \vec{r_2} e^{i|\vec{r}-\vec{r_2}|} + \vec{r_3}^{\,i|\vec{r}-\vec{r_3}|}\right) \cdot \vec{\delta r}$$

For the amplitude not to change, this must be, as a complex number, perpendicular to

$$e^{i|\vec{r}-\vec{r_1}|} + e^{i|\vec{r}-\vec{r_2}|} + e^{i|\vec{r}-\vec{r_3}|}$$

Note that

$$e^{i|\vec{r}-\vec{r_1}|} + e^{i|\vec{r}-\vec{r_2}|} + e^{i|\vec{r}-\vec{r_3}|} = 0$$

corresponds to the global minimum, which obviously cannot be a local maxima, so we assume it is nonzero. Then, the perpendicularity criterion is equivalent to

$$\frac{i\left(\vec{r_1} e^{i|\vec{r}-\vec{r_1}|} + \vec{r_2} e^{i|\vec{r}-\vec{r_2}|} + \vec{r_3}^{\,i|\vec{r}-\vec{r_3}|}\right) \cdot \vec{\delta r}}{e^{i|\vec{r}-\vec{r_1}|} + e^{i|\vec{r}-\vec{r_2}|} + e^{i|\vec{r}-\vec{r_3}|}}$$

being a purely complex number, or

$$\frac{\left(\vec{r_1}e^{i|\vec{r}-\vec{r_1}|}+\vec{r_2}e^{i|\vec{r}-\vec{r_2}|}+\vec{r_3}^{\,i|\vec{r}-\vec{r_3}|}\right)\cdot\vec{\delta r}}{e^{i|\vec{r}-\vec{r_1}|}+e^{i|\vec{r}-\vec{r_2}|}+e^{i|\vec{r}-\vec{r_3}|}}$$

being real. Now imagine

$$\frac{\left(\vec{r_1}e^{i|\vec{r}-\vec{r_1}|}+\vec{r_2}e^{i|\vec{r}-\vec{r_2}|}+\vec{r_3}^{\,i|\vec{r}-\vec{r_3}|}\right)}{e^{i|\vec{r}-\vec{r_1}|}+e^{i|\vec{r}-\vec{r_2}|}+e^{i|\vec{r}-\vec{r_3}|}}$$

as a vector with complex coefficients. For this to always have a real output when taken dot product with $\vec{\delta r}$, it is necessary and sufficient to have the vector

$$\frac{\left(\vec{r_1}e^{i|\vec{r}-\vec{r_1}|}+\vec{r_2}e^{i|\vec{r}-\vec{r_2}|}+\vec{r_3}^{\,i|\vec{r}-\vec{r_3}|}\right)}{e^{i|\vec{r}-\vec{r_1}|}+e^{i|\vec{r}-\vec{r_2}|}+e^{i|\vec{r}-\vec{r_3}|}}$$

having real coefficients (obviously it is sufficient, and it is necessary, by taking $\vec{\delta r}=(\delta r,0)$ and $(0,\delta r)$).

Consider the imaginary part of the vector. Let the

$$\frac{e^{i|\vec{r}-\vec{r_i}|}}{e^{i|\vec{r}-\vec{r_1}|}+e^{i|\vec{r}-\vec{r_2}|}+e^{i|\vec{r}-\vec{r_3}|}}$$

be $a_i+b_i i$. Note that $b_1+b_2+b_3=0$ (since those the sum of these three complex numbers is 1). We equivalently need the complex part $b_1\vec{r_1}+b_2\vec{r_2}+b_3\vec{r_3}$ to be $\vec{0}$, for the local maxima. In other words, $b_1(\vec{r_1}-\vec{r_3})+b_2(\vec{r_2}-\vec{r_3})=\vec{0}$, so equivalently we need either $b_1=b_2=b_3=0$ or for the light sources to be collinear.

However, looking at the photo, if the light sources were to be collinear, that axial line must be a symmetry axis line of the intensity maxima, but we do not see this. Therefore, we conclude that $b_1=b_2=b_3=0$, or that

$$\frac{e^{i|\vec{r}-\vec{r_i}|}}{e^{i|\vec{r}-\vec{r_1}|}+e^{i|\vec{r}-\vec{r_2}|}+e^{i|\vec{r}-\vec{r_3}|}}$$

are all real. Dividing them by each other, we see that each

$$e^{i|\vec{r}-\vec{r_i}|}$$

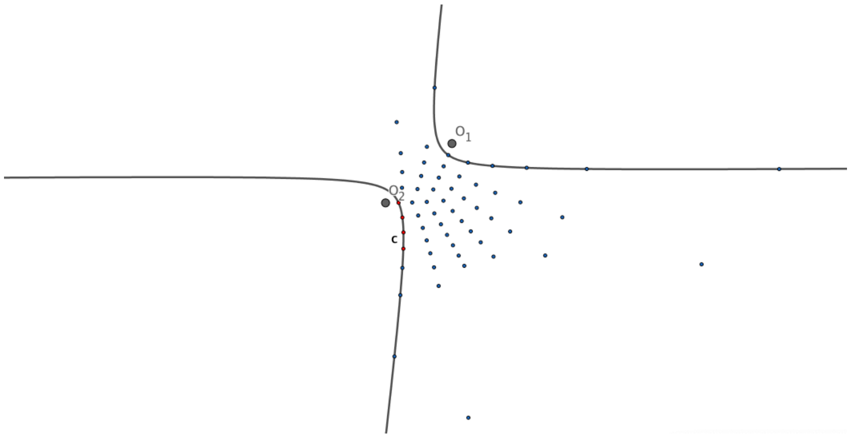
is real, so each pair of light sources is in phase.

To answer the question of this section, no, all the local maxima must exactly be the points where all the sources are in phase! This allows us to finally go to the GeoGebra diagram and so some geometry.

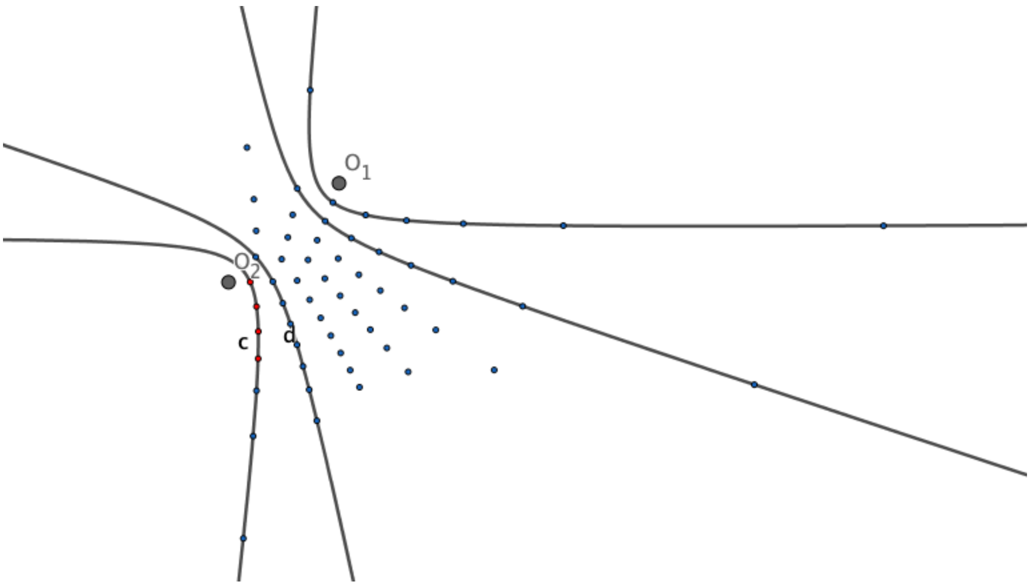
§2 Geometry

We want the locus of points that are some difference of distances $k\lambda$ away from P_1 and P_2. This is a set of hyperbolas. Now, if we think of the set of hyperbolas between P_1 and P_2 as $\mathcal{H}_{12}$, then the photo shows the intersections of $\mathcal{H}_{12}$, $\mathcal{H}_{23}$, $\mathcal{H}_{31}$. Thus, our goal is simple. We find the focus of the hyperbolas.

§2.1 Identifying a hyperbola



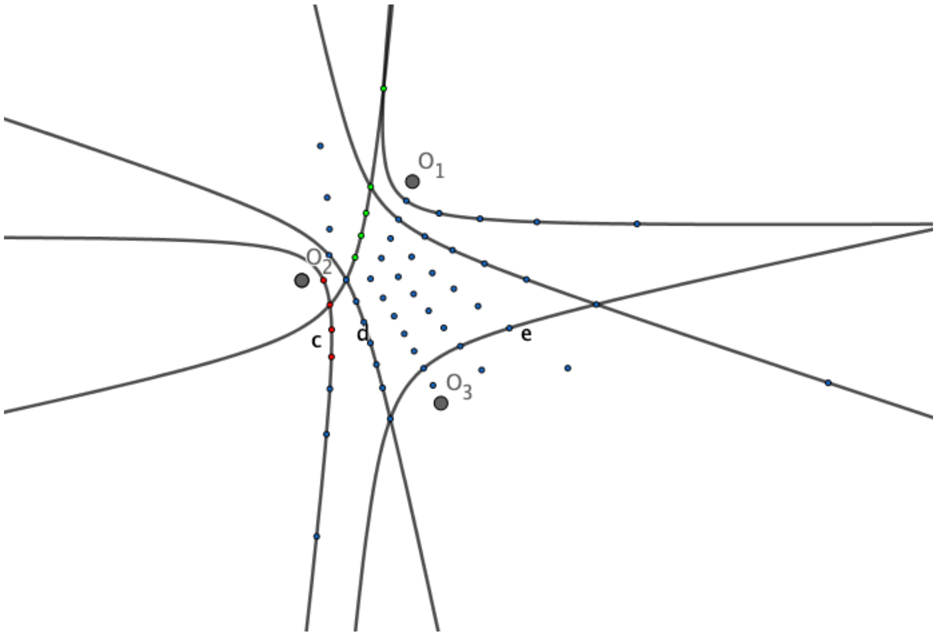
My first guess at a hyperbola was the five red points. Using the 5 points to conic tool of GeoGebra, I drew this hyperbola. It fits a set of other points! This is good confirmation that this is indeed a hyperbola we want. Then, using the Focus command, I get the focal points O_1 and O_2, at $(-0.04152, 3.33821)$ and $(-6.00000, -2.00000)$ respectively. I use the next set of hyperbola to chck this is correct.



It is! So, we already identified two of the light sources.

§3 Finding the other point

Consider the third hyperbola as shown in the diagram below, found by the green points.



This gives the two foci, O_2 and O_3. O_2 coincides as $(-6.00000, -2.00000)$ as we found earlier, and O_3 is $(1.49717, -8.61759)$.

§4 Answer

The light sources are positioned at $(-6.00000, -2.00000)$ and $(1.49717, -8.61759)$ and $(-0.04152, 3.33821)$

§5 Extra: Finding the wavelength

To do this, we will consider the phase differences of the hyperbolas we mentioned (this will need the distance tool). These are 4.00000 and 6.00000 and 6.00000, thus giving λ as 2.00000 (since we are already considering the nearest hyperbolas, it cannot be smaller).

§6 Comments

Really nice problem, I actually never thought about the phase differences in the way of hyperbolas.