Physicscup 2023 - Problem No 5

Marinus Lehmann

28.03.2022

1 Concepts

Of the interference pattern of three light sources all the points where the intensity has a local maximum are given. For the light sources three assumptions are made:

- They emit homogenously. For each source its electromagnetic radiation field only depends on the geometric distance to the source.
- The emit mutually coherent light, which is the basic for having a global interference pattern.
- The decrease of intensity (in 3 dimensions $\frac{1}{r^2}$) is neglected.

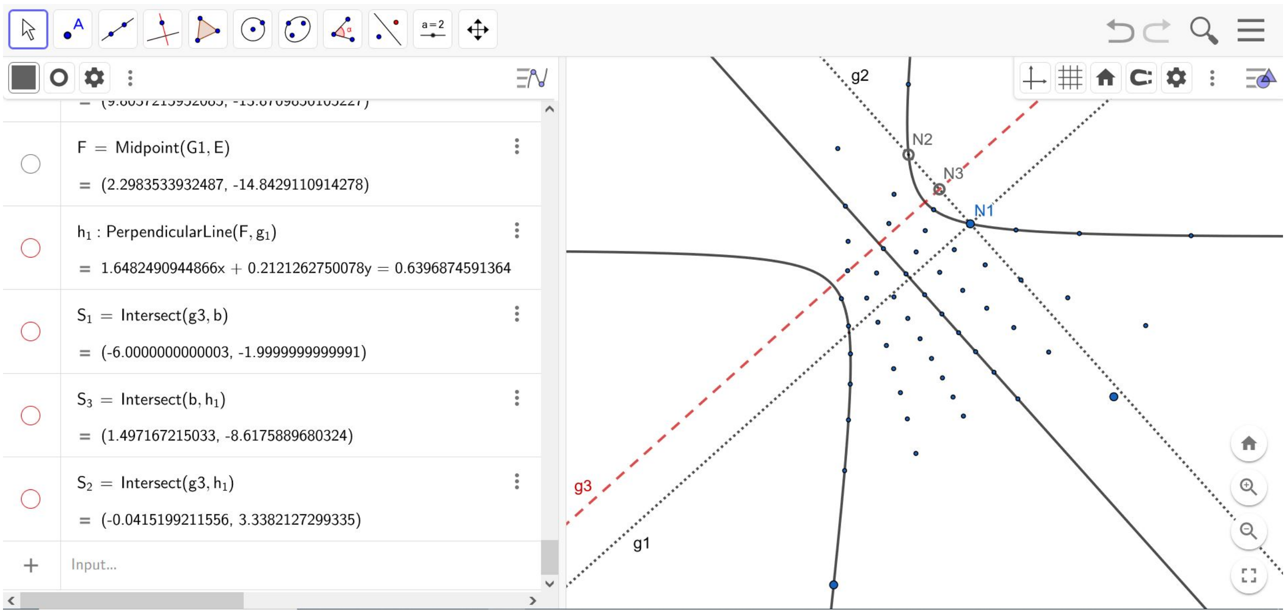
In general intensity maxima mean that the light is interfering in a constructive way. Due to the third assumption the mathematics gets far easier. Considering a decrease of intensity would mean the search of local maxima of the intensity distribution which has to in general done via multidimensional analysis ($\boldsymbol{\nabla} I = 0$ as main criteria). Here it is easier - as known from school - constructive interference happens when the phase of the waves differ by a multiple of 2π. This is of course equivalent to the statement that the phase shift between each two of the sources is $2\pi n, n \in \mathbb{Z}$. Or equivalent to: The difference in the path difference is a multiple of the wavelength.
This statement makes clear that the problem can be divided. First the interference pattern of two sources has to be found. This is not complicated. Constructive interference arises if the path difference $\Delta s_{12} = ||\vec{x} - \vec{s_1}||_2 - ||\vec{x} - \vec{s_2}||_2 = \sqrt{(x - x_{s1})^2 + (y - y_{s1})^2} - \sqrt{(x - x_{s2})^2 + (y - y_{s2})^2} = 2\pi n, n \in \mathbb{Z}$. For each n this means a constant path difference and the curve that fulfills this condition is well known - it is a hyperbola. The sources are here exactly the foci of the hyperbola. One can also verify this result by calculating and plotting the intensity field **??**.
Second the intensity maxima of the three sources are the points where the path difference between all three pairs is a multiple of 2π. This is only fulfilled at the points, where the interference hyperbolas intersect. Indeed for theory it is only necessary to consider two hyperbola arrays, because $\Delta s_{23} = \Delta s_{13} - \Delta s_{12}$ holds.

2 Construction

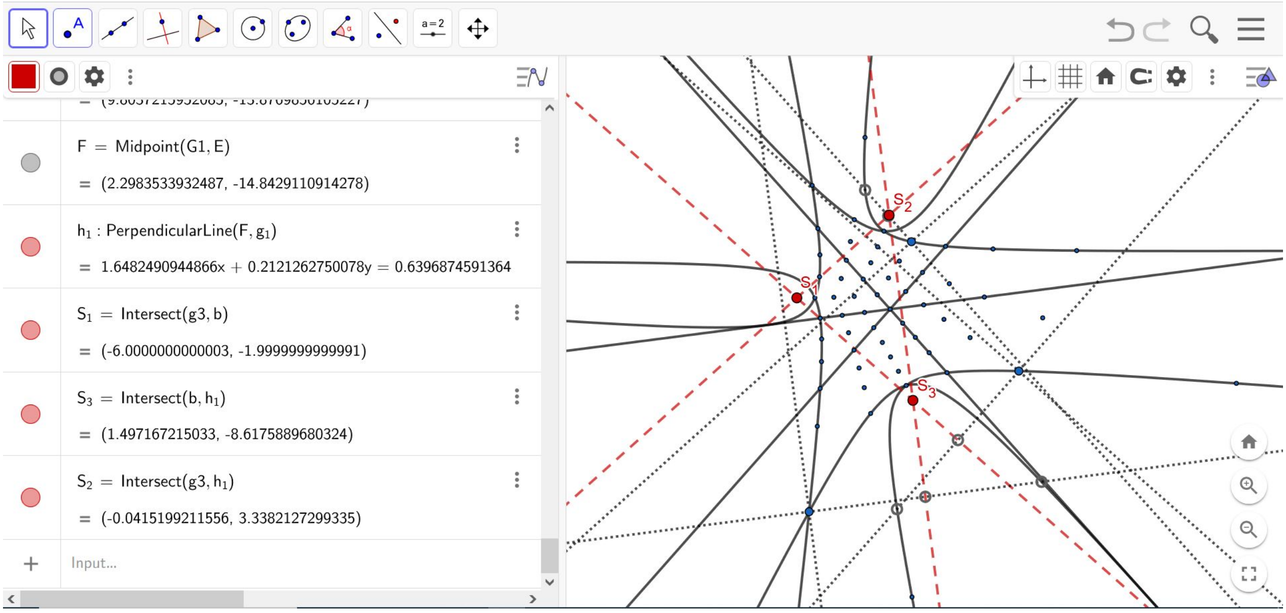
After understanding the basic concepts it is now possible to construct the sources in GeoGebra. First one has to find the underlying structure - the hyperbolas on which the maxima are lined up. The construction can be done by the tool Conic through 5 point-sünder the ellipsis symbol. As expected there are three arrays of hyperbolas **??**.
Especially interesting is the degenerated hyperbola in the middle which is simple a straight line. For each array of hyperbolas one has to find the two symmetry axes. One is already known, but the other one through the foci has to be constructed. This can be done in 5 steps needing only of each array only one hyperbola and the straight line (degenerated one), it will be called middle line"here. The five steps are:

- Take a point (N1) on the hyperbola and construct the orthogonal line (g1) with respect to the middle line.
- Create a second line (g2) through this point perpendicular to the line of step one.
- Mark the intersection point (N2) of the hyperbola and the line of step two.
- Create the middle point N3 of N1 and N2.
- Construct a straight line g3 which goes through N3 and is orthogonal to g2.

In GeoGebra it looks like this:



These five steps are the same for all of the three arrays of interference hyperbolas. What is still missing are the foci themselves. But for each focus it is known that it has to be on the symmetry axis of the hyperbolas as just constructed. As there are two such lines, the foci are simply the intersection points. So after $3 \cdot 5 + 3 = 18$ steps all the foci are found and the picture looks like this: As the foci are identical with the sources all



three are found.

3 Discussion and Appendix

In this part the intensity field is discussed, also considering the effects arising when assumption three does not hold.
Let us first have a view onto the intensity field under the given assumptions. The exact mathematics behind interference is the linear combination of the electromagnetic fields. The intensity is then the square of the amplitude averaged over time, because light has

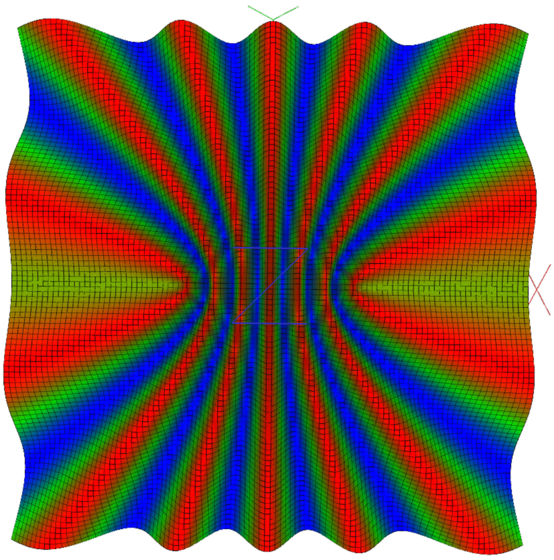
a very high frequency we cannot see time-resolved:

$$
\begin{aligned}
I_{No5} &= \bar{E}^2 \\
E_{No5} &= \sum_{i=1}^{3} \cos(\omega \cdot t - k \cdot ||\vec{x} - \vec{s_i}||_2) = \sum_{i=1}^{3} \cos\left(\omega \cdot t - \frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right) \\
&= \sum_{i=1}^{3} \cos(\omega \cdot t) \cdot \cos\left(-\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right) - \sin(\omega \cdot t) \cdot \sin\left(-\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right) \\
&= \cos(\omega \cdot t) \cdot \left(\sum_{i=1}^{3} \cos\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)\right) + \sin(\omega \cdot t) \left(\sum_{i=1}^{3} \cdot \sin\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)\right) \\
I_{No5} &= \bar{E}^2 = \overline{\cos(\omega \cdot t)^2} \cdot \left(\sum_{i=1}^{3} \cos\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)\right)^2 + \overline{\sin(\omega \cdot t)^2} \left(\sum_{i=1}^{3} \cdot \sin\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)\right. \\
&+ \overline{\cos(\omega \cdot t) \sin(\omega \cdot t)} \cdot \left(\sum_{i=1}^{3} \cos\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)\right) \left(\sum_{i=1}^{3} \sin\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)\right) \\
&= \frac{1}{2} \left[\left(\sum_{i=1}^{3} \cos\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)\right)^2 + \left(\sum_{i=1}^{3} \sin\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)\right)^2\right]
\end{aligned}
$$

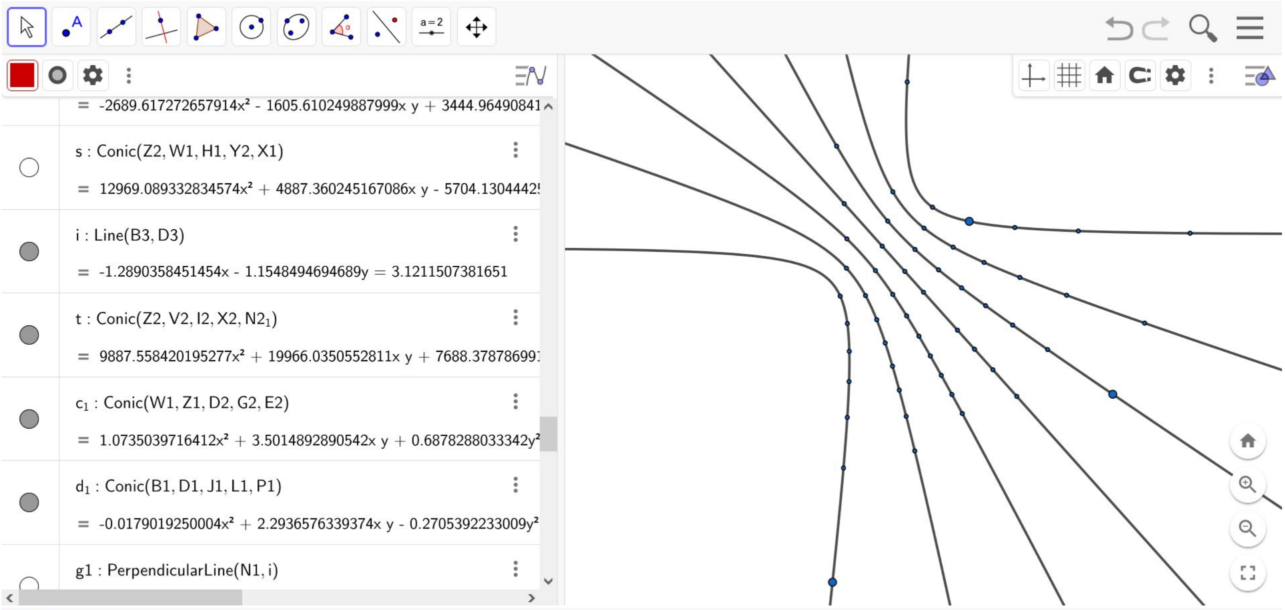
Plotting this formula for only two sources with $x_{s_1} = -10, x_{s_1} = 10$ and $y_{s_1} = y_{s_2} = 0$ the interference pattern looks like this: Indeed the interference maxima and minima have the shapes of hyperbolas.
As mentioned in the first part, with GeoGebra the hyperbolas can be constructed out of maxima, here they are:
If the third assumption does not hold, then the electric field is decreasing with $\frac{1}{r}$. The formula for the electric field and intensity are then modified to (same calculation with trigonometric identity and avering over time):

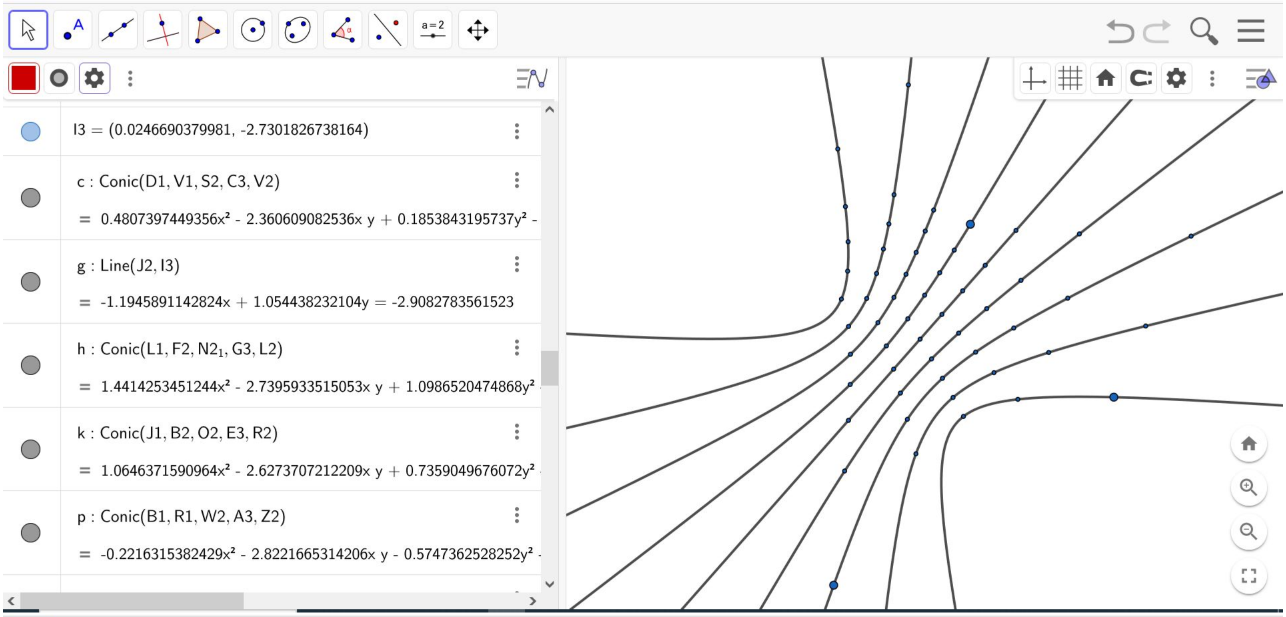
$$
\begin{aligned}
E_{gen} &= \sum_{i=1}^{3} \frac{\cos\left(\omega \cdot t - \frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)}{\sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}} \\
I_{gen} &= \bar{E}^2 \\
&= \frac{1}{2} \left[\left(\sum_{i=1}^{3} \frac{\cos\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)}{\sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}}\right)^2 + \left(\sum_{i=1}^{3} \frac{\sin\left(\frac{2\pi}{\lambda} \cdot \sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}\right)}{\sqrt{(x - x_{s_i})^2 + (y - y_{s_i})^2}}\right)^2\right]
\end{aligned}
$$

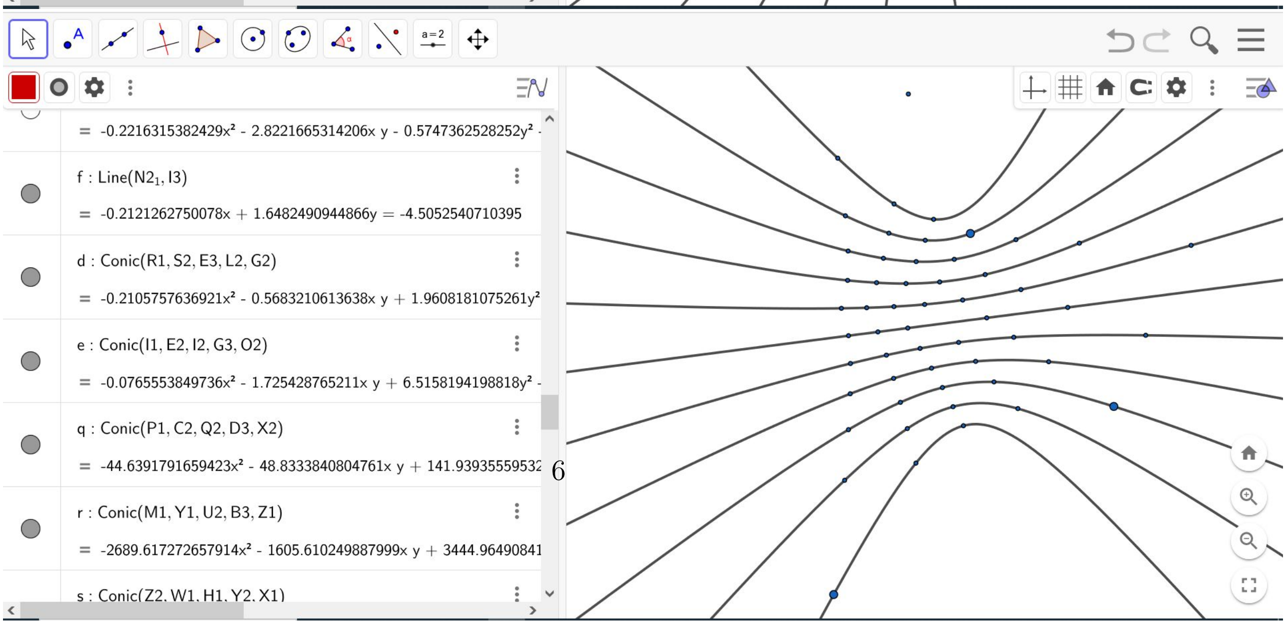
Far away from all sources the distances in first order the distance to all of them is similar. So one would expect that in the far field it is $I_{No5} \approx I_{gen} \cdot (x^2 + y^2)$. I proved this statement it by plotting the difference.
But in the problem the interference pattern is not in the far field of the sources. Here the properties of light are important. Except for the regions in the immediate vicinity of the sources there are local maxima and minima. Of course the difference between two orders of interference is on the scale of the wavelength. For light this scale is very short. If one imagines that the geometric configuration is on scale similar to the one on the picture ($\sim$ cm) then the wavelength is far shorter. So even great distortions in the phase mean only tiny corrections - on a level that is invisible for us. So the assumption three is very senseful as long as the wavelenght is short.



If the experiment would be with mechanical sources e.g. sound, then things could be different and it may be important to consider the near-field-effects of a distance-dependent intensity.


= -2689.617272657914x² - 1605.610249887999x y + 3444.96490841
s : Conic(Z2, W1, H1, Y2, X1)
= 12969.089332834574x² + 4887.360245167086x y - 5704.1304442
i : Line(B3, D3)
= -1.2890358451454x - 1.1548494694689y = 3.1211507381651
t : Conic(Z2, V2, I2, X2, N2₁)
= 9887.558420195277x² + 19966.0350552811x y + 7688.37878699
c₁ : Conic(W1, Z1, D2, G2, E2)
= 1.0735039716412x² + 3.5014892890542x y + 0.6878288033342y²
d₁ : Conic(B1, D1, J1, L1, P1)
= -0.0179019250004x² + 2.2936576339374x y - 0.2705392223009y²
g1 : PerpendicularLine(N1, i)


I3 = (0.0246690379981, -2.7301826738164)
c : Conic(D1, V1, S2, C3, V2)
= 0.4807397449356x² - 2.360609082536x y + 0.1853843195737y² -
g : Line(J2, I3)
= -1.1945891142824x + 1.054438232104y = -2.9082783561523
h : Conic(L1, F2, N2₁, G3, L2)
= 1.4414253451244x² - 2.7395933515053x y + 1.0986520474868y²
k : Conic(J1, B2, O2, E3, R2)
= 1.0646371590964x² - 2.6273707212209x y + 0.7359049676072y²
p : Conic(B1, R1, W2, A3, Z2)
= -0.2216315382429x² - 2.8221665314206x y - 0.5747362528252y²


= -0.2216315382429x² - 2.8221665314206x y - 0.5747362528252y²
f : Line(N2₁, I3)
= -0.2121262750078x + 1.6482490944866y = -4.5052540710395
d : Conic(R1, S2, E3, L2, G2)
= -0.2105757636921x² - 0.5683210613638x y + 1.9608181075261y²
e : Conic(I1, E2, I2, G3, O2)
= -0.0765553849736x² - 1.725428765211x y + 6.5158194198818y²
q : Conic(P1, C2, Q2, D3, X2)
= -44.6391791659423x² - 48.8333840804761x y + 141.93935559532
r : Conic(M1, Y1, U2, B3, Z1)
= -2689.617272657914x² - 1605.610249887999x y + 3444.96490841
s : Conic(Z2, W1, H1, Y2, X1)